New Binary Degree 2 BBP-type Formulas*

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Contents

1 Introduction

The obtention of a desired degree s BBP-type formulas proceeds in two stages

1. The polylogarithm constant of interest is first expressed as a linear combination of the real or imaginary parts of polylogarithms.

2. The following identities for the real and imaginary parts of a polylogarithm function are then employed to write each constituent term of the linear combination as a BBP-type formula and the indicated combination is then formed:

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$$\mathrm{Re}\,\mathrm{Li}_s\left[pe^{ix}\right] = \sum_{k=1}^{\infty} \frac{p^k \cos kx}{k^s} \tag{1}$$

$$\mathrm{Im}\,\mathrm{Li}_s\left[pe^{ix}\right] = \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k^s}\,, \tag{2}$$

for $p \in [0,1]$, $x \in \mathbb{R}$ and $s \in \mathbb{Z}^+$. In the above equations, Li is the notation for the polylogarithm function defined by

$$\mathrm{Li}_s[z] = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad |z| \le 1\,.$$

To accomplish the first stage, polylogarithm functional equations are evaluated at certain carefully chosen coordinates.

2 Generators of Degree 2 BBP-type Formulas

The dilogarithm reflection formula (Eq. A.2.1.7 of [1]) is

$$\frac{\pi^2}{6} - \ln x \ln(1-x) = \mathrm{Li}_2[x] + \mathrm{Li}_2[1-x]\,.$$

Putting $x = 1/2$ in the above formula gives the well-known result:

$$\frac{\pi^2}{12} - \frac{\ln^2 2}{2} = \mathrm{Li}_2\left[\frac{1}{2}\right] \tag{3}$$

A two-variable functional equation for dilogarithms, due to Kummer (Eq. A.2.1.19 of [1]) is

$$\begin{aligned}\mathrm{Li}_2\left[\frac{x(1-y)^2}{y(1-x)^2}\right] &= \mathrm{Li}_2\left[-\frac{x(1-y)}{(1-x)}\right] + \mathrm{Li}_2\left[-\frac{(1-y)}{y(1-x)}\right] \\ &\quad + \mathrm{Li}_2\left[\frac{x}{y}\frac{(1-y)}{(1-x)}\right] + \mathrm{Li}_2\left[\frac{1-y}{1-x}\right] + \frac{1}{2}\ln^2 y\,.\end{aligned} \tag{4}$$

Putting $x = -1$ and $y = 1/2$ in Kummer's formula gives

$$\ln^2 2 = 2\,\mathrm{Li}_2\left[-\frac{1}{8}\right] - 4\,\mathrm{Li}_2\left[\frac{1}{4}\right] - 4\,\mathrm{Li}_2\left[-\frac{1}{2}\right] \tag{5}$$

Putting $x = \exp(i\pi/3)$ and $y = 1/2$ in Kummer's formula gives

$$\pi^2 = 72\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{3}\right)\right] - 18\,\mathrm{Li}_2\left[\frac{1}{4}\right] \tag{6}$$

Putting $x = 1/2$ and $y = \exp(i\pi/2)$ in Kummer's formula gives

$$\begin{aligned}\ln^2 2 = 2\,\mathrm{Li}_2\left[-\frac{1}{4}\right] - 4\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] \\ - 4\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right]\end{aligned} \tag{7}$$

Putting $x = -1$ and $y = (1+i)/2$ in Kummer's formula and taking real and imaginary parts give

$$\begin{aligned}\frac{\pi \ln 2}{8} = 2\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] - 2\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\ - \mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\end{aligned} \tag{8}$$

and

$$\begin{aligned}\frac{\pi^2}{32} - \frac{1}{8}\ln^2 2 = 2\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] + 2\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\ - \mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\,.\end{aligned} \tag{9}$$

Another two-variable functional equation for dilogarithms, due to Abel (Eq. A.2.1.16 of [1]) is

$$\begin{aligned}\mathrm{Li}_2\left[\frac{x}{1-x}\cdot\frac{y}{1-y}\right] = \mathrm{Li}_2\left[\frac{x}{(1-y)}\right] + \mathrm{Li}_2\left[\frac{y}{(1-x)}\right] \\ - \mathrm{Li}_2\,[x] - \mathrm{Li}_2\,[y] - \ln(1-x)\ln(1-y)\,.\end{aligned} \tag{10}$$

Putting $x = i$ and $y = -1$ in Abels formula and taking real and imaginary parts, gives

$$\begin{aligned}\frac{5\pi^2}{48} - \frac{1}{2}\ln^2 2 = \mathrm{Re}\ \mathrm{Li}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right] \\ - \mathrm{Re}\ \mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] - \mathrm{Re}\ \mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right]\end{aligned} \tag{11}$$

and

$$\begin{aligned}\mathrm{G} - \frac{\pi \ln 2}{4} = \mathrm{Im}\ \mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] \\ + \mathrm{Im}\ \mathrm{Li}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right] - \mathrm{Im}\ \mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right]\end{aligned} \tag{12}$$

Putting $x = 1/2$, $y = \exp(i\pi/3)$ in Abels formula and taking the imaginary part, gives

$$5\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \pi\ln 2 = 6\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{3}\right)\right] \tag{13}$$

Putting $x = i = y$ in Abels formula and taking real and imaginary parts, gives

$$\frac{5\pi^2}{48} - \frac{\ln^2 2}{4} = \mathrm{Re}\;\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] - 2\,\mathrm{Re}\;\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] \tag{14}$$

and

$$2\,\mathrm{G} - \frac{\pi\ln 2}{4} = 2\,\mathrm{Im}\;\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] + \mathrm{Im}\;\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] \tag{15}$$

3 Base 2^{12} Formulas

Solving Eqs. (11) and (14) simultaneously, we find

$$\begin{aligned}\ln^2 2 = {} & 8\,\mathrm{Re}\;\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] - 4\,\mathrm{Re}\;\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] \\ & - 4\,\mathrm{Re}\;\mathrm{Li}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right]\end{aligned} \tag{16}$$

and

$$\begin{aligned}\pi^2 = {} & \frac{144}{5}\,\mathrm{Re}\;\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] - \frac{144}{5}\,\mathrm{Re}\;\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] \\ & - \frac{48}{5}\,\mathrm{Re}\;\mathrm{Li}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right]\end{aligned} \tag{17}$$

Solving Eqs. (12) and (15) simultaneously, we find

$$\mathrm{G} = 3\,\mathrm{Im}\;\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] - \mathrm{Im}\;\mathrm{Li}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right] \tag{18}$$

and

$$\begin{aligned}\pi\ln 2 = {} & 16\,\mathrm{Im}\;\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] - 4\,\mathrm{Im}\;\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] \\ & - 8\,\mathrm{ImLi}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right]\end{aligned} \tag{19}$$

3.1 BBP-type formula for π^2

3.2 BBP-type formula for $\ln^2 2$

3.3 BBP-type formula for $\pi \ln 2$

3.4 BBP-type formula for Catalan's Constant G

4 Base 2^{60} length 120 Formulas

Solving Eqs. (3) and (9) simultaneously, we have

$$\begin{aligned}\pi^2 &= 192\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] + 192\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\ &\quad -96\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 24\,\mathrm{Li}_2\left[\frac{1}{2}\right] \end{aligned} \tag{20}$$

and

$$\begin{aligned}\log^2 2 &= 32\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] + 32\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\ &\quad -16\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 6\,\mathrm{Li}_2\left[\frac{1}{2}\right] \end{aligned} \tag{21}$$

Using Eq. (8) in Eq. (12), we have

$$\begin{aligned}G &= 5\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] - \mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i3\pi}{4}\right)\right] \\ &\quad -3\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 2\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \end{aligned} \tag{22}$$

4.1 BBP-type formula for π^2

Writing each member of Eq. (20) as a base 2^{60}, length 120 series and forming the indicated linear combination leads to the binary BBP-type formula

$$
\begin{aligned}
\pi^2 \quad = \quad & \frac{3}{2^{54}} P(2, 2^{60}, 120, (0, -2^{58}, 3^2 \cdot 2^{58}, -3^2 \cdot 2^{57}, -5^2 \cdot 2^{56}, -2^{56}, 0, \\
& 7 \cdot 2^{55}, -3^2 \cdot 2^{55}, -2^{54}, 0, -3^3 \cdot 2^{53}, 0, -2^{52}, 7 \cdot 2^{51}, 7 \cdot 2^{51}, 0, -2^{50}, 0, 2^{53}, \\
& 3^2 \cdot 2^{49}, -2^{48}, 0, 5^2 \cdot 2^{47}, 5^2 \cdot 2^{46}, -2^{46}, 3^2 \cdot 2^{46}, -3^2 \cdot 2^{45}, 0, -2^{44}, 0, 7 \cdot 2^{43}, \\
& -3^2 \cdot 2^{43}, -2^{42}, -5^2 \cdot 2^{41}, -3^3 \cdot 2^{41}, 0, -2^{40}, -3^2 \cdot 2^{40}, -3^2 \cdot 2^{40}, 0, -2^{38}, 0, \\
& -3^2 \cdot 2^{37}, -7 \cdot 2^{36}, -2^{36}, 0, 5^2 \cdot 2^{35}, 0, -2^{34}, 3^2 \cdot 2^{34}, -3^2 \cdot 2^{33}, 0, -2^{32}, 5^2 \cdot 2^{31}, \\
& 7 \cdot 2^{31}, -3^2 \cdot 2^{31}, -2^{30}, 0, -2^{30}, 0, -2^{28}, -3^2 \cdot 2^{28}, 7 \cdot 2^{27}, 5^2 \cdot 2^{26}, -2^{26}, 0, \\
& -3^2 \cdot 2^{25}, 3^2 \cdot 2^{25}, -2^{24}, 0, 5^2 \cdot 2^{23}, 0, -2^{22}, -7 \cdot 2^{21}, -3^2 \cdot 2^{21}, 0, -2^{20}, 0, \\
& -3^2 \cdot 2^{20}, -3^2 \cdot 2^{19}, -2^{18}, 0, -3^3 \cdot 2^{17}, -5^2 \cdot 2^{16}, -2^{16}, -3^2 \cdot 2^{16}, 7 \cdot 2^{15}, 0, \\
& -2^{14}, 0, -3^2 \cdot 2^{13}, 3^2 \cdot 2^{13}, -2^{12}, 5^2 \cdot 2^{11}, 5^2 \cdot 2^{11}, 0, -2^{10}, 3^2 \cdot 2^{10}, 2^{13}, \\
& 0, -2^8, 0, 7 \cdot 2^7, 7 \cdot 2^6, -2^6, 0, -3^3 \cdot 2^5, 0, -2^4, -3^2 \cdot 2^4, 7 \cdot 2^3, 0, -2^2, \\
& -5^2 \cdot 2, -3^2 \cdot 2, 3^2 \cdot 2, -1, 0, 0)) \qquad (23)
\end{aligned}
$$

4.2 BBP-type formula for $\log^2 2$

Writing each member of Eq. (21) as a base 2^{60}, length 120 series and forming the indicated linear combination leads to the binary BBP-type formula

$$
\begin{aligned}
\log^2 2 \quad = \quad & \frac{1}{2^{57}} P(2, 2^{60}, 120, (0, -3 \cdot 2^{59}, 3^2 \cdot 2^{60}, -19 \cdot 2^{58}, -5^2 \cdot 2^{58}, -3 \cdot 2^{57}, 0, \\
& 13 \cdot 2^{56}, -3^2 \cdot 2^{57}, -3 \cdot 2^{55}, 0, -5 \cdot 11 \cdot 2^{54}, 0, -3 \cdot 2^{53}, 7 \cdot 2^{53}, 13 \cdot 2^{52}, 0, -3 \cdot 2^{51}, 0, \\
& 31 \cdot 2^{50}, 3^2 \cdot 2^{51}, -3 \cdot 2^{49}, 0, 7^2 \cdot 2^{48}, 5^2 \cdot 2^{48}, -3 \cdot 2^{47}, 3^2 \cdot 2^{48}, -19 \cdot 2^{46}, 0, \\
& -3 \cdot 2^{45}, 0, 13 \cdot 2^{44}, -3^2 \cdot 2^{45}, -3 \cdot 2^{43}, -5^2 \cdot 2^{43}, -5 \cdot 11 \cdot 2^{42}, 0, -3 \cdot 2^{41}, -3^2 \cdot 2^{42}, \\
& -37 \cdot 2^{40}, 0, -3 \cdot 2^{39}, 0, -19 \cdot 2^{38}, -7 \cdot 2^{38}, -3 \cdot 2^{37}, 0, 7^2 \cdot 2^{36}, 0, -3 \cdot 2^{35}, 3^2 \cdot 2^{36}, \\
& -19 \cdot 2^{34}, 0, -3 \cdot 2^{33}, 5^2 \cdot 2^{33}, 13 \cdot 2^{32}, -3^2 \cdot 2^{33}, -3 \cdot 2^{31}, 0, -5 \cdot 2^{30}, 0, -3 \cdot 2^{29}, \\
& -3^2 \cdot 2^{30}, 13 \cdot 2^{28}, 5^2 \cdot 2^{28}, -3 \cdot 2^{27}, 0, -19 \cdot 2^{26}, 3^2 \cdot 2^{27}, -3 \cdot 2^{25}, 0, 7^2 \cdot 2^{24}, 0, \\
& -3 \cdot 2^{23}, -7 \cdot 2^{23}, -19 \cdot 2^{22}, 0, -3 \cdot 2^{21}, 0, -37 \cdot 2^{20}, -3^2 \cdot 2^{21}, -3 \cdot 2^{19}, 0, -5 \cdot 11 \cdot 2^{18}, \\
& -5^2 \cdot 2^{18}, -3 \cdot 2^{17}, -3^2 \cdot 2^{18}, 13 \cdot 2^{16}, 0, -3 \cdot 2^{15}, 0, -19 \cdot 2^{14}, 3^2 \cdot 2^{15}, -3 \cdot 2^{13}, \\
& 5^2 \cdot 2^{13}, 7^2 \cdot 2^{12}, 0, -3 \cdot 2^{11}, 3^2 \cdot 2^{12}, 31 \cdot 2^{10}, 0, -3 \cdot 2^9, 0, 13 \cdot 2^8, 7 \cdot 2^8, -3 \cdot 2^7, \\
& 0, -5 \cdot 11 \cdot 2^6, 0, -3 \cdot 2^5, -3^2 \cdot 2^6, 13 \cdot 2^4, 0, -3 \cdot 2^3, -5^2 \cdot 2^3, -19 \cdot 2^2, \\
& 3^2 \cdot 2^3, -3 \cdot 2, 0, -1)) \qquad (24)
\end{aligned}
$$

4.3 BBP-type formula for $\pi \ln 2$

Writing each member of Eq. (8) as a base 2^{60}, length 120 series and forming the indicated linear combination leads to the binary BBP-type formula

$$
\begin{aligned}
\pi \log 2 \quad = \quad & \frac{1}{2^{55}} P(2, 2^{60}, 120, (0, 2^{60}, -3^2 \cdot 2^{57}, 0, -5^2 \cdot 2^{55}, -13 \cdot 2^{56}, 0, 0, -3^2 \cdot 2^{54}, -17 \cdot 2^{53}, \\
& 0, 0, 0, -2^{54}, -7 \cdot 2^{50}, 0, 0, 13 \cdot 2^{50}, 0, 0, 3^2 \cdot 2^{48}, -2^{50}, 0, 0, 5^2 \cdot 2^{45}, 2^{48}, -3^2 \cdot 2^{45}, 0, 0, \\
& -2^{43}, 0, 0, -3^2 \cdot 2^{42}, 2^{44}, 5^2 \cdot 2^{40}, 0, 0, -2^{42}, 3^2 \cdot 2^{39}, 0, 0, 13 \cdot 2^{38}, 0, 0, -7 \cdot 2^{35}, -2^{38}, \\
& 0, 0, 0, -17 \cdot 2^{33}, -3^2 \cdot 2^{33}, 0, 0, -13 \cdot 2^{32}, -5^2 \cdot 2^{30}, 0, -3^2 \cdot 2^{30}, 2^{32}, 0, 0, 0, -2^{30}, \\
& 3^2 \cdot 2^{27}, 0, 5^2 \cdot 2^{25}, 13 \cdot 2^{26}, 0, 0, 3^2 \cdot 2^{24}, 17 \cdot 2^{23}, 0, 0, 0, 2^{24}, 7 \cdot 2^{20}, 0, 0, -13 \cdot 2^{20}, \\
& 0, 0, -3^2 \cdot 2^{18}, 2^{20}, 0, 0, -5^2 \cdot 2^{15}, -2^{18}, 3^2 \cdot 2^{15}, 0, 0, 2^{13}, 0, 0, 3^2 \cdot 2^{12}, -2^{14}, -5^2 \cdot 2^{10}, \\
& 0, 0, 2^{12}, -3^2 \cdot 2^{9}, 0, 0, -13 \cdot 2^{8}, 0, 0, 7 \cdot 2^{5}, 2^{8}, 0, 0, 0, 17 \cdot 2^{3}, 3^2 \cdot 2^{3}, 0, 0, \\
& 13 \cdot 2^{2}, 5^2, 0, 3^2, -2^2, 0, 0)) \qquad (25)
\end{aligned}
$$

4.4 BBP-type formula for Catalan's Constant G

Finally, writing each member of Eq. (22) as a base 2^{60}, length 120 series and forming the indicated linear combination leads to the binary BBP-type formula

$$
\begin{aligned}
G \quad = \quad & \frac{1}{2^{60}} P(2, 2^{60}, 120, (-2^{59}, 3 \cdot 7 \cdot 2^{59}, -7 \cdot 2^{60}, 0, -7^2 \cdot 2^{57}, -3 \cdot 2^{61}, 2^{56}, 0, \\
& -7 \cdot 2^{57}, -29 \cdot 2^{55}, -2^{54}, 0, 2^{53}, -3 \cdot 7 \cdot 2^{53}, -11 \cdot 2^{53}, 0, -2^{51}, 3 \cdot 2^{55}, -2^{50}, 0, \\
& 7 \cdot 2^{51}, -3 \cdot 7 \cdot 2^{49}, 2^{48}, 0, 7^2 \cdot 2^{47}, 3 \cdot 7 \cdot 2^{47}, -7 \cdot 2^{48}, 0, 2^{45}, 2^{46}, 2^{44}, 0, \\
& -7 \cdot 2^{45}, 3 \cdot 7 \cdot 2^{43}, 7^2 \cdot 2^{42}, 0, 2^{41}, -3 \cdot 7 \cdot 2^{41}, 7 \cdot 2^{42}, 0, -2^{39}, 3 \cdot 2^{43}, -2^{38}, \\
& 0, -11 \cdot 2^{38}, -3 \cdot 7 \cdot 2^{37}, 2^{36}, 0, -2^{35}, -29 \cdot 2^{35}, -7 \cdot 2^{36}, 0, 2^{33}, -3 \cdot 2^{37}, -7^2 \cdot 2^{32}, \\
& 0, -7 \cdot 2^{33}, 3 \cdot 7 \cdot 2^{31}, -2^{30}, 0, 2^{29}, -3 \cdot 7 \cdot 2^{29}, 7 \cdot 2^{30}, 0, 7^2 \cdot 2^{27}, 3 \cdot 2^{31}, -2^{26}, 0, \\
& 7 \cdot 2^{27}, 29 \cdot 2^{25}, 2^{24}, 0, -2^{23}, 3 \cdot 7 \cdot 2^{23}, 11 \cdot 2^{23}, 0, 2^{21}, -3 \cdot 2^{25}, 2^{20}, 0, -7 \cdot 2^{21}, \\
& 3 \cdot 7 \cdot 2^{19}, -2^{18}, 0, -7^2 \cdot 2^{17}, -3 \cdot 7 \cdot 2^{17}, 7 \cdot 2^{18}, 0, -2^{15}, -2^{16}, -2^{14}, 0, 7 \cdot 2^{15}, \\
& -3 \cdot 7 \cdot 2^{13}, -7^2 \cdot 2^{12}, 0, -2^{11}, 3 \cdot 7 \cdot 2^{11}, -7 \cdot 2^{12}, 0, 2^{9}, -3 \cdot 2^{13}, 2^{8}, 0, 11 \cdot 2^{8}, \\
& 3 \cdot 7 \cdot 2^{7}, -2^{6}, 0, 2^{5}, 29 \cdot 2^{5}, 7 \cdot 2^{6}, 0, -2^{3}, 3 \cdot 2^{7}, 7^2 \cdot 2^{2}, 0, \\
& 7 \cdot 2^{3}, -3 \cdot 7 \cdot 2, 1, 0)) \qquad (26)
\end{aligned}
$$

5 A Degree 2 Zero Relation

Eliminating $\pi \log 2$ between Eq. (8) and Eq. (19) gives the identity

$$
\begin{aligned}
0 \quad = \quad & 5 \,\mathrm{Im}\,\mathrm{Li}_2 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2} \right) \right] - 4 \,\mathrm{Im}\,\mathrm{Li}_2 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i3\pi}{4} \right) \right] \\
& -2 \,\mathrm{Im}\,\mathrm{Li}_2 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right] - 2 \,\mathrm{Im}\,\mathrm{Li}_2 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right] \qquad (27)
\end{aligned}
$$

from which we get, immediately, the binary BBP-type zero relation

$$
\begin{aligned}
0 \quad = \quad & P(2, 2^{60}, 120, (-2^{59}, 3 \cdot 2^{60}, -11 \cdot 2^{57}, 0, -23 \cdot 2^{56}, -3 \cdot 7 \cdot 2^{56}, 2^{56}, 0, -11 \cdot 2^{54}, -13 \cdot 2^{54}, \\
& -2^{54}, 0, 2^{53}, -3 \cdot 2^{54}, -7 \cdot 2^{52}, 0, -2^{51}, 3 \cdot 7 \cdot 2^{50}, -2^{50}, 0, 11 \cdot 2^{48}, -3 \cdot 2^{50}, 2^{48}, 0, \\
& 23 \cdot 2^{46}, 3 \cdot 2^{48}, -11 \cdot 2^{45}, 0, 2^{45}, 2^{46}, 2^{44}, 0, -11 \cdot 2^{42}, 3 \cdot 2^{44}, 23 \cdot 2^{41}, 0, 2^{41}, -3 \cdot 2^{42}, \\
& 11 \cdot 2^{39}, 0, -2^{39}, 3 \cdot 7 \cdot 2^{38}, -2^{38}, 0, -7 \cdot 2^{37}, -3 \cdot 2^{38}, 2^{36}, 0, -2^{35}, -13 \cdot 2^{34}, -11 \cdot 2^{33}, 0, \\
& 2^{33}, -3 \cdot 7 \cdot 2^{32}, -23 \cdot 2^{31}, 0, -11 \cdot 2^{30}, 3 \cdot 2^{32}, -2^{30}, 0, 2^{29}, -3 \cdot 2^{30}, 11 \cdot 2^{27}, 0, 23 \cdot 2^{26}, \\
& 3 \cdot 7 \cdot 2^{26}, -2^{26}, 0, 11 \cdot 2^{24}, 13 \cdot 2^{24}, 2^{24}, 0, -2^{23}, 3 \cdot 2^{24}, 7 \cdot 2^{22}, 0, 2^{21}, -3 \cdot 7 \cdot 2^{20}, 2^{20}, \\
& 0, -11 \cdot 2^{18}, 3 \cdot 2^{20}, -2^{18}, 0, -23 \cdot 2^{16}, -3 \cdot 2^{18}, 11 \cdot 2^{15}, 0, -2^{15}, -2^{16}, -2^{14}, 0, 11 \cdot 2^{12}, \\
& -3 \cdot 2^{14}, -23 \cdot 2^{11}, 0, -2^{11}, 3 \cdot 2^{12}, -11 \cdot 2^{9}, 0, 2^{9}, -3 \cdot 7 \cdot 2^{8}, 2^{8}, 0, 7 \cdot 2^{7}, 3 \cdot 2^{8}, -2^{6}, 0, \\
& 2^{5}, 13 \cdot 2^{4}, 11 \cdot 2^{3}, 0, -2^{3}, 3 \cdot 7 \cdot 2^{2}, 23 \cdot 2, 0, 11, -3 \cdot 2^{2}, 1, 0)) \qquad (28)
\end{aligned}
$$

6 Conclusion

References

[1] Leonard Lewin. *Polylogarithms and associated functions.* Elsevier North Holland Inc., 1981.