New Binary Degree 3 Digit Extraction (BBP-type) Formulas*

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Abstract

No *proved* explicit Digit Extraction BBP-type formulas exist in the literature for π^3 and $\pi \log^2 2$. In this paper we present newly discovered BBP-type formulas for these and other trilogarithm constants. New binary degree 3 zero relations are also proved.

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1 Introduction

BBP-type formulas are formulas of the form

$$\alpha = \sum_{k=0}^{\infty} 1/b^k \sum_{j=1}^{l} a_j/(kl+j)^s$$

where s, b, l and a_j are integers, and α is some constant. Formulas of this type were first introduced in a 1996 paper [1], where a formula of this type for π was given. Such formulas have the remarkable property that they permit one to calculate base-b digits of the constant α beginning at an arbitrary starting position, by means of a simple algorithm that requires almost no memory and (depending on how many digits are required) without the need for multiple-precision arithmetic software [2]. Such formulas also have intriguing connections to the age-old problem of understanding why the digits of various transcendental constants appear "normal" – each string of m-long digits appears, in the limit, with frequency $1/b^m$ [3, 4, 5, 2].

No *proved* explicit Digit Extraction BBP-type formulas are known for π^3 and $\pi \log^2 2$. In what follows, we now present, together with their proofs, new binary degree 3 BBP-type formulas for these and the remaining three trilogarithm constants.

2 Generators of Degree 3 BBP-type formulas

A functional equation for trilogarithms (Eq. A.2.6.10 of [6]) reads

$$\begin{aligned} \mathrm{Li}_3\left[\frac{1-x}{1+x}\right] - \mathrm{Li}_3\left[\frac{x-1}{x+1}\right] &= 2\,\mathrm{Li}_3\left[1-x\right] + 2\,\mathrm{Li}_3\left[\frac{1}{1+x}\right] \\ &\quad - \frac{1}{2}\,\mathrm{Li}_3\left[1-x^2\right] - \frac{7}{4}\,\zeta(3) \\ &\quad + \frac{\pi^2}{6}\log(1+x) - \frac{1}{3}\log^3(1+x)\,. \end{aligned} \tag{1}$$

In the above equation and throughout this paper, Li is the notation for the polylogarithm function defined by

$$\mathrm{Li}_s[z] = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad |z| \le 1,\ s \in \mathbb{Z}^+\,.$$

The use of $x = 1$ in the functional equation (1) gives the well-known formula

$$\frac{7}{8}\zeta(3) - \frac{\pi^2 \log 2}{12} + \frac{\log^3 2}{6} = \mathrm{Li}_3\left[\frac{1}{2}\right]\,. \tag{2}$$

Another functional identity for trilogarithms (Eq. A.2.6.11 of [6]) is

$$
\begin{aligned}
&\mathrm{Li}_3\left[\frac{x(1-y)^2}{y(1-x)^2}\right] + \mathrm{Li}_3\left[xy\right] + \mathrm{Li}_3\left[\frac{x}{y}\right] \\
&- 2\mathrm{Li}_3\left[\frac{x(1-y)}{y(1-x)}\right] - 2\mathrm{Li}_3\left[\frac{x(1-y)}{(x-1)}\right] - 2\mathrm{Li}_3\left[\frac{1-y}{1-x}\right] \\
&- 2\mathrm{Li}_3\left[\frac{(1-y)}{y(x-1)}\right] - 2\mathrm{Li}_3\left[x\right] - 2\mathrm{Li}_3\left[y\right] + 2\zeta(3) \\
&\qquad = \log^2 y \log\left(\frac{1-y}{1-x}\right) - \frac{1}{3}\pi^2 \log y - \frac{1}{3}\log^3 y\,.
\end{aligned}
\tag{3}
$$

The use of $x = -1$, $y = i$ in the above equation gives

$$
\frac{35}{8}\zeta(3) - \frac{5\pi^2 \log 2}{24} + \frac{1}{6}\log^3 2 = 8\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\,. \tag{4}
$$

Plugging $x = -i$, $y = 1 - i$ in Eq. (3) and taking real and imaginary parts gives

$$
\begin{aligned}
&\frac{7}{16}\zeta(3) + \frac{5\pi^2 \log 2}{192} - \frac{7\log^3 2}{48} \\
&\quad = 2\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + \frac{9}{4}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] \\
&\qquad - \mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
\frac{3\pi \log^2 2}{32} - \frac{9\pi^3}{128} &= \mathrm{Im}\,\mathrm{Li}_3\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
&\quad + \frac{9}{4}\,\mathrm{Im}\,\mathrm{Li}_3\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] - 6\,\mathrm{Im}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\,.
\end{aligned}
\tag{6}
$$

Yet another functional equation for trilogarithms (Eq. 6.96 p 174 of [6]) is

$$
\begin{aligned}
\mathrm{Li}_3\left[\frac{x(1-y)^2}{y(1-x)^2}\right] &= 2\mathrm{Li}_3\left[\frac{-x(1-y)}{(1-x)}\right] + 2\mathrm{Li}_3\left[\frac{x(1-y)}{y(1-x)}\right] \\
&+ 2\mathrm{Li}_3\left[\frac{-y(1-x)}{(1-y)}\right] + 2\mathrm{Li}_3\left[\frac{1-x}{1-y}\right] + 2\mathrm{Li}_3\left[x\right] \\
&+ 2\mathrm{Li}_3\left[y\right] - 2\mathrm{Li}_3\left[xy\right] - \mathrm{Li}_3\left[\frac{x}{y}\right] - 2\zeta(3) \\
&+ \frac{\pi^2}{3}\log\left(\frac{1-y}{1-x}\right) + \log(-y)\log^2(1-y) \\
&- \log^2(-y)\log(1-x) - \frac{1}{3}\log^3\left(\frac{1-y}{1-x}\right) \\
&+ \frac{1}{3}\log^3\left(\frac{-y(1-x)}{1-y}\right) + \frac{1}{3}\log^3\left(-\frac{1-y}{y}\right) \\
&- \frac{1}{3}\log^3(1-y)\,.
\end{aligned} \tag{7}
$$

Using $x=-1$, $y=1+i$ in Eq. (7), simplifying and taking real and imaginary parts gives

$$
\begin{aligned}
&\frac{7}{2}\zeta(3) - \frac{15\pi^2\log 2}{64} + \frac{5\log^3 2}{16} \\
&= 4\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + 4\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
&+ \frac{7}{2}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] - \mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
&\frac{13\pi^3}{128} - \frac{7\pi\log^2 2}{32} \\
&= 4\,\mathrm{Im}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 4\,\mathrm{Im}\,\mathrm{Li}_3\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
&+ \frac{7}{2}\,\mathrm{Im}\,\mathrm{Li}_3\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] - \mathrm{Im}\,\mathrm{Li}_3\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right].
\end{aligned} \tag{9}
$$

Using $x=-1$, $y=1/2$ in Eq. (7) gives the identity

$$
7\,\zeta(3) - \pi^2\log 2 + 3\,\log^3 2 = 9\,\mathrm{L}_3\left[\frac{1}{4}\right] - 2\,\mathrm{L}_3\left[-\frac{1}{8}\right]. \tag{10}
$$

We are now ready to derive formulas for π^3, $\pi\log^2 2$, $\zeta(3)$, $\pi^2\log 2$ and $\log^3 2$. In order to save space, we will give the formulas using the compact P-notation [2]:

$$P(s,b,l,A) \equiv \sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{l} \frac{a_j}{(kl+j)^s}\,,$$

where s, b and l are integers, and $A = (a_1, a_2, \dots, a_n)$ is a vector of integers.

3 Base 2^{12} Formulas

Solving Eqs. (2), (4) and (10) simultaneously, we find the BBP-ready identities:

$$\begin{aligned}\zeta(3) \quad &= \quad \frac{128}{21}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - \frac{88}{21}\,\mathrm{Li}_3\left[\frac{1}{2}\right] + \frac{12}{7}\,\mathrm{Li}_3\left[\frac{1}{4}\right] \\ &\qquad -\frac{8}{21}\,\mathrm{Li}_3\left[-\frac{1}{8}\right]\end{aligned} \tag{11}$$

$$\begin{aligned}\pi^2\log 2 \quad &= \quad \frac{320}{3}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - \frac{328}{3}\,\mathrm{Li}_3\left[\frac{1}{2}\right] + 48\,\mathrm{Li}_3\left[\frac{1}{4}\right] \\ &\qquad -\frac{32}{3}\,\mathrm{Li}_3\left[-\frac{1}{8}\right]\end{aligned} \tag{12}$$

and

$$\begin{aligned}\log^3 2 \quad &= \quad \frac{64}{3}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - \frac{80}{3}\,\mathrm{Li}_3\left[\frac{1}{2}\right] + 15\,\mathrm{Li}_3\left[\frac{1}{4}\right] \\ &\qquad -\frac{10}{3}\,\mathrm{Li}_3\left[-\frac{1}{8}\right]\,.\end{aligned} \tag{13}$$

3.1 Binary BBP-type Formula for $\zeta(3)$

Using the identity

$$\mathrm{Re}\ \mathrm{Li}_s\left[pe^{iq}\right] = \sum_{k=1}^{\infty} \frac{p^k \cos(kq)}{k^s} \qquad p \in [0,1], q \in [0,2\pi], s \in \mathbb{Z}^+\,,$$

to write each term of the Right Hand Side of Eq. (11) and forming the indicated linear combination leads to

$$\begin{aligned}\zeta(3) \quad &= \quad \frac{1}{21\cdot 2^5} P(3, 2^{12}, 24, (2^{11}, -11\cdot 2^{10}, -2^{10}, 23\cdot 2^9, -2^9, 2^{12}, \\ &\quad 2^8, 27\cdot 2^7, 2^7, -11\cdot 2^6, -2^6, -2^7, -2^5, -11\cdot 2^4, 2^4, 27\cdot 2^3, 2^3, \\ &\quad 2^6, -2^2, 23\cdot 2^1, -2^1, -11, 1, 0))\end{aligned} \tag{14}$$

Valid variants of this formula for $\zeta(3)$ can be obtained by adding rational multiples of the zero relation (29) to Eq. (14).

3.2 Binary BBP-type Formula for $\pi^2 \log 2$

The polylogarithm identity Eq. (12) gives the formula

$$\begin{aligned}\pi^2 \log 2 &= \frac{1}{3 \cdot 2^6} P(3, 2^{12}, 24, (5 \cdot 2^{11}, -41 \cdot 2^{11}, -5 \cdot 2^{10}, 49 \cdot 2^{11}, \\ &-5 \cdot 2^9, 67 \cdot 2^9, 5 \cdot 2^8, 27 \cdot 2^{10}, 5 \cdot 2^7, -41 \cdot 2^7, -5 \cdot 2^6, -5 \cdot 2^7, \\ &-5 \cdot 2^5, -41 \cdot 2^5, 5 \cdot 2^4, 27 \cdot 2^6, 5 \cdot 2^3, 67 \cdot 2^3, -5 \cdot 2^2, 49 \cdot 2^3, \\ &-5 \cdot 2^1, -41 \cdot 2^1, 5, 0))\,. \end{aligned} \tag{15}$$

Valid variants of this formula for $\pi^2 \log 2$ can be obtained by adding rational multiples of the zero relation (29).

3.3 Binary BBP-type Formula for $\log^3 2$

Finally, identity Eq. (13) leads to the binary BBP-type formula:

$$\begin{aligned}\log^3 2 &= \frac{1}{3 \cdot 2^8} P(3, 2^{12}, 24, (2^{13}, -5 \cdot 2^{14}, -2^{12}, 17 \cdot 2^{13}, -2^{11}, \\ &5 \cdot 19 \cdot 2^9, 2^{10}, 9 \cdot 2^{12}, 2^9, -5 \cdot 2^{10}, -2^8, 2^6, -2^7, -5 \cdot 2^8, 2^6, \\ &9 \cdot 2^8, 2^5, 5 \cdot 19 \cdot 2^3, -2^4, 17 \cdot 2^5, -2^3, -5 \cdot 2^4, 2^2, 9))\,. \end{aligned} \tag{16}$$

Valid variants of this formula for $\log^3 2$ can be obtained by adding rational multiples of the zero relation (29).

4 Base 2^{60} Formulas

4.1 Binary BBP-type Formula for π^3

Eliminating $\pi \log^2 2$ between Eqs. (6) and (9), we have the following BBP-ready formula for π^3:

$$\begin{aligned}\pi^3 = 16\, \mathrm{Im}\, \mathrm{Li}_3 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] + 160\, \mathrm{Im}\, \mathrm{Li}_3 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] \\ + \frac{80}{3}\, \mathrm{Im}\, \mathrm{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] - 140\, \mathrm{Im}\, \mathrm{Li}_3 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right)\right]\,. \end{aligned} \tag{17}$$

Using the identity

$$\mathrm{Im}\, \mathrm{Li}_s \left[p e^{iq}\right] = \sum_{k=1}^{\infty} \frac{p^k \sin(kq)}{k^s} \quad p \in [0, 1], q \in [0, 2\pi], s \in \mathbb{Z}^+\,,$$

each term of Eq. (17) can be written as a BBP-type formula of length 120 and base 2^{60}. Doing this judiciously and combining according to Eq. (17), we find the binary BBP-type formula for π^3 to be:

$$
\begin{aligned}
\pi^3 = 5/2^{55} P \big(&3, 2^{60}, 120, (2^{59}, -3\cdot 2^{60}, 11\cdot 2^{57}, 0, 23\cdot 2^{56}, 3\cdot 7\cdot 2^{56},\\
&-2^{56}, 0, 11\cdot 2^{54}, 13\cdot 2^{54}, 2^{54}, 0, -2^{53}, 3\cdot 2^{54},\\
&7\cdot 2^{52}, 0, 2^{51}, -3\cdot 7\cdot 2^{50}, 2^{50}, 0, -11\cdot 2^{48}, 3\cdot 2^{50}, -2^{48}, 0,\\
&-23\cdot 2^{46}, -3\cdot 2^{48}, 11\cdot 2^{45}, 0, -2^{45}, -2^{46}, -2^{44}, 0,\\
&11\cdot 2^{42}, -3\cdot 2^{44}, -23\cdot 2^{41}, 0, -2^{41}, 3\cdot 2^{42}, -11\cdot 2^{39}, 0,\\
&2^{39}, -3\cdot 7\cdot 2^{38}, 2^{38}, 0, 7\cdot 2^{37}, 3\cdot 2^{38}, -2^{36}, 0, 2^{35}, 13\cdot 2^{34}, 11\cdot 2^{33}, 0,\\
&-2^{33}, 3\cdot 7\cdot 2^{32}, 23\cdot 2^{31}, 0, 11\cdot 2^{30}, -3\cdot 2^{32}, 2^{30}, 0, -2^{29}, 3\cdot 2^{30}, -11\cdot 2^{27}, 0,\\
&-23\cdot 2^{26}, -3\cdot 7\cdot 2^{26}, 2^{26}, 0, -11\cdot 2^{24}, -13\cdot 2^{24}, -2^{24}, 0, 2^{23}, -3\cdot 2^{24},\\
&-7\cdot 2^{22}, 0, -2^{21}, 3\cdot 7\cdot 2^{20}, -2^{20}, 0, 11\cdot 2^{18}, -3\cdot 2^{20}, 2^{18}, 0, 23\cdot 2^{16},\\
&3\cdot 2^{18}, -11\cdot 2^{15}, 0, 2^{15}, 2^{16}, 2^{14}, 0, -11\cdot 2^{12}, 3\cdot 2^{14}, 23\cdot 2^{11}, 0,\\
&2^{11}, -3\cdot 2^{12}, 11\cdot 2^{9}, 0, -2^{9}, 3\cdot 7\cdot 2^{8},\\
&-2^{8}, 0, -7\cdot 2^{7}, -3\cdot 2^{8}, 2^{6}, 0, -2^{5},\\
&-13\cdot 2^{4}, -11\cdot 2^{3}, 0, 2^{3}, -3\cdot 7\cdot 2^{2}, -23\cdot 2, 0, -11, 3\cdot 2^{2}, -1, 0)\big)\ .
\end{aligned}
\tag{18}
$$

Valid variants of this formula for π^3 can be obtained by adding rational linear combinations of the zero relations (31) and (33).

4.2 Binary BBP-type Formula for $\pi \log^2 2$

Eliminating π^3 between Eqs. (6) and (9), we have the following BBP-ready formula for $\pi \log^2 2$:

$$
\begin{aligned}
\pi \log^2 2 = 12 \,\mathrm{Im}\,\mathrm{Li}_3 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] + 56 \,\mathrm{Im}\,\mathrm{Li}_3 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] \\
+ \frac{92}{3} \,\mathrm{Im}\,\mathrm{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] - 81 \,\mathrm{Im}\,\mathrm{Li}_3 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right)\right] .
\end{aligned}
\tag{19}
$$

Writing each term of Eq. (19) as a base 2^{60}, length 120 series as in the previous section and forming the indicated linear combination, we find the binary BBP-type formula for $\pi \log^2 2$ as

$$\pi \log^2 2 = 1/2^{57} P(3, 2^{60}, 120, (7 \cdot 2^{59}, -37 \cdot 2^{60}, 13 \cdot 17 \cdot 2^{57}, 0, 19^2 \cdot 2^{56},$$
$$5 \cdot 71 \cdot 2^{56}, -7 \cdot 2^{56}, 0, 13 \cdot 17 \cdot 2^{54}, 227 \cdot 2^{54}, 7 \cdot 2^{54}, 0, -7 \cdot 2^{53}, 37 \cdot 2^{54},$$
$$7 \cdot 11 \cdot 2^{52}, 0, 7 \cdot 2^{51}, -5 \cdot 71 \cdot 2^{50}, 7 \cdot 2^{50}, 0, -13 \cdot 17 \cdot 2^{48}, 37 \cdot 2^{50}, -7 \cdot 2^{48}, 0, -19^2 \cdot 2^{46},$$
$$-37 \cdot 2^{48}, 13 \cdot 17 \cdot 2^{45}, 0, -7 \cdot 2^{45}, -5 \cdot 2^{46}, -7 \cdot 2^{44}, 0, 13 \cdot 17 \cdot 2^{42},$$
$$-37 \cdot 2^{44}, -19^2 \cdot 2^{41}, 0, -7 \cdot 2^{41}, 37 \cdot 2^{42}, -13 \cdot 17 \cdot 2^{39}, 0, 7 \cdot 2^{39}, -5 \cdot 71 \cdot 2^{38}, 7 \cdot 2^{38},$$
$$0, 7 \cdot 11 \cdot 2^{37}, 37 \cdot 2^{38}, -7 \cdot 2^{36}, 0, 7 \cdot 2^{35}, 227 \cdot 2^{34}, 13 \cdot 17 \cdot 2^{33}, 0, -7 \cdot 2^{33},$$
$$5 \cdot 71 \cdot 2^{32}, 19^2 \cdot 2^{31}, 0, 13 \cdot 17 \cdot 2^{30}, -37 \cdot 2^{32}, 7 \cdot 2^{30},$$
$$0, -7 \cdot 2^{29}, 37 \cdot 2^{30}, -13 \cdot 17 \cdot 2^{27}, 0, -19^2 \cdot 2^{26}, -5 \cdot 71 \cdot 2^{26}, 7 \cdot 2^{26}, 0, -13 \cdot 17 \cdot 2^{24},$$
$$-227 \cdot 2^{24}, -7 \cdot 2^{24}, 0, 7 \cdot 2^{23}, -37 \cdot 2^{24}, -7 \cdot 11 \cdot 2^{22}, 0,$$
$$-7 \cdot 2^{21}, 5 \cdot 71 \cdot 2^{20}, -7 \cdot 2^{20}, 0, 13 \cdot 17 \cdot 2^{18}, -37 \cdot 2^{20}, 7 \cdot 2^{18}, 0, 19^2 \cdot 2^{16}, 37 \cdot 2^{18}, -13 \cdot 17 \cdot 2^{15},$$
$$0, 7 \cdot 2^{15}, 5 \cdot 2^{16}, 7 \cdot 2^{14}, 0, -13 \cdot 17 \cdot 2^{12}, 37 \cdot 2^{14}, 19^2 \cdot 2^{11}, 0, 7 \cdot 2^{11}, -37 \cdot 2^{12}, 13 \cdot 17 \cdot 2^{9},$$
$$0, -7 \cdot 2^{9}, 5 \cdot 71 \cdot 2^{8}, -7 \cdot 2^{8}, 0, -7 \cdot 11 \cdot 2^{7}, -37 \cdot 2^{8}, 7 \cdot 2^{6}, 0, -7 \cdot 2^{5},$$
$$-227 \cdot 2^{4}, -13 \cdot 17 \cdot 2^{3}, 0, 7 \cdot 2^{3}, -5 \cdot 71 \cdot 2^{2}, -19^2 \cdot 2, 0, -17 \cdot 13, 37 \cdot 2^{2}, -7, 0)) . \tag{20}$$

Valid variants of this formula for $\pi \log^2 2$ can be obtained by adding rational linear combinations of the zero relations (31) and (33).

4.3 Binary BBP-type Formula for $\zeta(3)$

Eliminating $\pi^2 \log 2$ and $\log^3 2$ between Eqs. (4), (5) and (8) we find

$$\begin{aligned}\zeta(3) = {} & \frac{16}{7} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] + \frac{32}{7} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] \\ & - \frac{48}{7} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] - \frac{92}{7} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right)\right] .\end{aligned} \tag{21}$$

Using the identity

$$\operatorname{Re} \operatorname{Li}_s \left[p e^{iq}\right] = \sum_{k=1}^{\infty} \frac{p^k \cos(kq)}{k^s} \quad p \in [0, 1], q \in [0, 2\pi], s \in \mathbb{Z}^+ ,$$

each term of Eq. (21) can be written as a BBP-type formula of length 120 and base 2^{60}. Doing this judiciously and combining according to Eq. (21), we find the binary BBP-type formula for $\zeta(3)$ to be:

$$
\begin{aligned}
\zeta(3) = 1/7/2^{55} P(3, 2^{60}, 120, (&2^{59}, 0, -83 \cdot 2^{57}, 11 \cdot 2^{59}, 3 \cdot 41 \cdot 2^{56},\\
&0, 2^{56}, -11 \cdot 2^{57}, 83 \cdot 2^{54}, 0, -2^{54}, 5^3 \cdot 2^{53}, -2^{53}, 0, -3 \cdot 7 \cdot 2^{52},\\
&-11 \cdot 2^{53}, 2^{51}, 0, -2^{50}, -3^4 \cdot 2^{49}, -83 \cdot 2^{48}, 0, 2^{48},\\
&-5^3 \cdot 2^{47}, -3 \cdot 41 \cdot 2^{46}, 0, -83 \cdot 2^{45}, 11 \cdot 2^{47},\\
&-2^{45}, 0, 2^{44}, -11 \cdot 2^{45}, 83 \cdot 2^{42}, 0, 3 \cdot 41 \cdot 2^{41}, 5^3 \cdot 2^{41}, -2^{41}, 0, 83 \cdot 2^{39},\\
&3^4 \cdot 2^{39}, 2^{39}, 0, -2^{38}, 11 \cdot 2^{39}, 3 \cdot 7 \cdot 2^{37}, 0, 2^{36}, -5^3 \cdot 2^{35}, 2^{35},\\
&0, -83 \cdot 2^{33}, 11 \cdot 2^{35}, -2^{33}, 0, -3 \cdot 41 \cdot 2^{31}, -11 \cdot 2^{33}, 83 \cdot 2^{30},\\
&0, -2^{30}, 0, -2^{29}, 0, 83 \cdot 2^{27}, -11 \cdot 2^{29}, -3 \cdot 41 \cdot 2^{26}, 0, -2^{26}, 11 \cdot 2^{27}, -83 \cdot 2^{24},\\
&0, 2^{24}, -5^3 \cdot 2^{23}, 2^{23}, 0, 3 \cdot 7 \cdot 2^{22}, 11 \cdot 2^{23}, -2^{21},\\
&0, 2^{20}, 3^4 \cdot 2^{19}, 83 \cdot 2^{18}, 0, -2^{18}, 5^3 \cdot 2^{17},\\
&3 \cdot 41 \cdot 2^{16}, 0, 83 \cdot 2^{15}, -11 \cdot 2^{17}, 2^{15}, 0, -2^{14}, 11 \cdot 2^{15},\\
&-83 \cdot 2^{12}, 0, -3 \cdot 41 \cdot 2^{11}, -5^3 \cdot 2^{11}, 2^{11}, 0,\\
&-83 \cdot 2^{9}, -3^4 \cdot 2^{9}, -2^{9}, 0, 2^{8}, -11 \cdot 2^{9}, -3 \cdot 7 \cdot 2^{7}, 0, -2^{6},\\
&5^3 \cdot 2^{5}, -2^{5}, 0, 83 \cdot 2^{3}, -11 \cdot 2^{5}, 2^{3}, 0, 3 \cdot 41 \cdot 2, 11 \cdot 2^{3}, -83, 0, 1, 0))
\end{aligned} \tag{22}
$$

Valid variants of this formula for $\zeta(3)$ can be obtained by adding rational linear combinations of the zero relations (31) and (33).

4.4 Binary BBP-type Formula for $\pi^2 \log 2$

Eliminating $\zeta(3)$ and $\log^3 2$ between Eqs. (4), (5) and (8) we find

$$
\begin{aligned}
\pi^2 \log 2 = \frac{312}{5} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right] + \frac{336}{5} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right] \\
- \frac{904}{5} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right] - \frac{1866}{5} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right) \right] .
\end{aligned} \tag{23}
$$

Writing each member of Eq. (23) as a base 2^{60}, length 120 series as in the previous section, we find the binary BBP-type formula for $\pi^2 \log 2$ as

$$
\begin{aligned}
\pi^2 \log 2 = 3/5/2^{56} P(3, 2^{60}, 120, (&7 \cdot 2^{59}, 0, \\
&- 1031 \cdot 2^{57}, 19 \cdot 2^{62}, 3^2 \cdot 179 \cdot 2^{56}, 0, 7 \cdot 2^{56}, -19 \cdot 2^{60}, \\
&1031 \cdot 2^{54}, 0, -7 \cdot 2^{54}, 5^3 \cdot 13 \cdot 2^{53}, -7 \cdot 2^{53}, \\
&0, -3^3 \cdot 11 \cdot 2^{52}, -19 \cdot 2^{56}, 7 \cdot 2^{51}, 0, -7 \cdot 2^{50}, \\
&- 3^2 \cdot 113 \cdot 2^{49}, -1031 \cdot 2^{48}, 0, 7 \cdot 2^{48}, -5^3 \cdot 13 \cdot 2^{47}, \\
&- 3^2 \cdot 179 \cdot 2^{46}, 0, -1031 \cdot 2^{45}, 19 \cdot 2^{50}, -7 \cdot 2^{45}, 0, \\
&7 \cdot 2^{44}, -19 \cdot 2^{48}, 1031 \cdot 2^{42}, 0, 3^2 \cdot 179 \cdot 2^{41}, \\
&5^3 \cdot 13 \cdot 2^{41}, -7 \cdot 2^{41}, 0, 1031 \cdot 2^{39}, \\
&3^2 \cdot 113 \cdot 2^{39}, 7 \cdot 2^{39}, 0, -7 \cdot 2^{38}, 19 \cdot 2^{42}, \\
&3^3 \cdot 11 \cdot 2^{37}, 0, 7 \cdot 2^{36}, -5^3 \cdot 13 \cdot 2^{35}, 7 \cdot 2^{35}, \\
&0, -1031 \cdot 2^{33}, 19 \cdot 2^{38}, -7 \cdot 2^{33}, 0, -3^2 \cdot 179 \cdot 2^{31}, \\
&- 19 \cdot 2^{36}, 1031 \cdot 2^{30}, 0, -7 \cdot 2^{30}, 0, -7 \cdot 2^{29}, \\
&0, 1031 \cdot 2^{27}, -19 \cdot 2^{32}, -3^2 \cdot 179 \cdot 2^{26}, \\
&0, -7 \cdot 2^{26}, 19 \cdot 2^{30}, -1031 \cdot 2^{24}, 0, 7 \cdot 2^{24}, \\
&- 5^3 \cdot 13 \cdot 2^{23}, 7 \cdot 2^{23}, 0, 3^3 \cdot 11 \cdot 2^{22}, 19 \cdot 2^{26}, \\
&- 7 \cdot 2^{21}, 0, 7 \cdot 2^{20}, 3^2 \cdot 113 \cdot 2^{19}, \\
&1031 \cdot 2^{18}, 0, -7 \cdot 2^{18}, 5^3 \cdot 13 \cdot 2^{17}, \\
&3^2 \cdot 179 \cdot 2^{16}, 0, 1031 \cdot 2^{15}, \\
&- 19 \cdot 2^{20}, 7 \cdot 2^{15}, 0, -7 \cdot 2^{14}, 19 \cdot 2^{18}, \\
&- 1031 \cdot 2^{12}, 0, -3^2 \cdot 179 \cdot 2^{11}, \\
&- 5^3 \cdot 13 \cdot 2^{11}, 7 \cdot 2^{11}, 0, -1031 \cdot 2^9, -3^2 \cdot 113 \cdot 2^9, \\
&- 7 \cdot 2^9, 0, 7 \cdot 2^8, -19 \cdot 2^{12}, -3^3 \cdot 11 \cdot 2^7, 0, \\
&- 7 \cdot 2^6, 5^3 \cdot 13 \cdot 2^5, -7 \cdot 2^5, 0, 1031 \cdot 2^3, \\
&- 19 \cdot 2^8, 7 \cdot 2^3, 0, 3^2 \cdot 179 \cdot 2, 19 \cdot 2^6, -1031, 0, 7, 0))
\end{aligned} \tag{24}
$$

Valid variants of this formula for $\pi^2 \log 2$ can be obtained by adding rational linear combinations of the zero relations (31) and (33).

4.5 Binary BBP-type Formula for $\log^3 2$

Elimination of $\pi^2 \log 2$ and $\zeta(3)$ between Eqs. (4), (5) and (8) gives

$$
\begin{aligned}
\log^3 2 = 18 \,\mathrm{Re}\,\mathrm{Li}_3 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] + 12 \,\mathrm{Re}\,\mathrm{Li}_3 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] \\
- 46 \,\mathrm{Re}\,\mathrm{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] - \frac{243}{2} \,\mathrm{Re}\,\mathrm{Li}_3 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right)\right] .
\end{aligned} \tag{25}
$$

Expressing each member of (25) as a base 2^{60}, length 120 series and forming the linear combination according to Eq. (25), we obtain a BBP-type formula for $\log^3 2$:

$$\begin{aligned}\log^3 2 = 3/2^{58} P(3, 2^{60}, 120, (&2^{59}, 0, -11 \cdot 19 \cdot 2^{57}, 5 \cdot 2^{62}, 373 \cdot 2^{56}, 0, 2^{56}, -5 \cdot 2^{60},\\
&11 \cdot 19 \cdot 2^{54}, 0, -2^{54}, 367 \cdot 2^{53}, -2^{53}, 0, -83 \cdot 2^{52}, -5 \cdot 2^{56}, 2^{51}, 0,\\
&-2^{50}, -5 \cdot 43 \cdot 2^{49}, -11 \cdot 19 \cdot 2^{48}, 0, 2^{48}, -367 \cdot 2^{47}, -373 \cdot 2^{46}, 0,\\
&-11 \cdot 19 \cdot 2^{45}, 5 \cdot 2^{50}, -2^{45}, 0, 2^{44}, -5 \cdot 2^{48}, 11 \cdot 19 \cdot 2^{42}, 0, 373 \cdot 2^{41},\\
&367 \cdot 2^{41}, -2^{41}, 0, 11 \cdot 19 \cdot 2^{39}, 5 \cdot 43 \cdot 2^{39}, 2^{39}, 0, -2^{38}, 5 \cdot 2^{42},\\
&83 \cdot 2^{37}, 0, 2^{36}, -367 \cdot 2^{35}, 2^{35}, 0, -11 \cdot 19 \cdot 2^{33}, 5 \cdot 2^{38}, -2^{33}, 0, -373 \cdot 2^{31},\\
&-5 \cdot 2^{36}, 11 \cdot 19 \cdot 2^{30}, 0, -2^{30}, -2^{32}, -2^{29}, 0, 11 \cdot 19 \cdot 2^{27}, -5 \cdot 2^{32}, -373 \cdot 2^{26}, 0,\\
&-2^{26}, 5 \cdot 2^{30}, -11 \cdot 19 \cdot 2^{24}, 0, 2^{24}, -367 \cdot 2^{23}, 2^{23}, 0, 83 \cdot 2^{22}, 5 \cdot 2^{26}, -2^{21},\\
&0, 2^{20}, 5 \cdot 43 \cdot 2^{19}, 11 \cdot 19 \cdot 2^{18}, 0, -2^{18}, 367 \cdot 2^{17}, 373 \cdot 2^{16}, 0, 11 \cdot 19 \cdot 2^{15},\\
&-5 \cdot 2^{20}, 2^{15}, 0, -2^{14}, 5 \cdot 2^{18}, -11 \cdot 19 \cdot 2^{12}, 0, -373 \cdot 2^{11}, -367 \cdot 2^{11}, 2^{11}, 0,\\
&-11 \cdot 19 \cdot 2^{9}, -5 \cdot 43 \cdot 2^{9}, -2^{9}, 0, 2^{8}, -5 \cdot 2^{12}, -83 \cdot 2^{7}, 0, -2^{6}, 367 \cdot 2^{5}, -2^{5}, 0,\\
&11 \cdot 19 \cdot 2^{3}, -5 \cdot 2^{8}, 2^{3}, 0, 373 \cdot 2, 5 \cdot 2^{6}, -19 \cdot 11, 0, 1, 2^{2})) \,.\end{aligned} \tag{26}$$

Valid variants of this formula for $\log^3 2$ can be obtained by adding rational linear combinations of the zero relations (31) and (33).

5 Zero Relations

5.1 Base 2^{12} Zero Relation

Eliminating $\pi^2 \log 2$ and $\log^3 2$ between Eqs. (4), (5) and (10), we find the BBP-ready identity:

$$\begin{aligned}\zeta(3) &= \frac{1584}{133} \operatorname{Re} \mathrm{Li}_3 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right)\right] + \frac{576}{133} \operatorname{Re} \mathrm{Li}_3 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right]\\
&\quad - \frac{704}{133} \operatorname{Re} \mathrm{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] + \frac{360}{133} \mathrm{Li}_3 \left[\frac{1}{4}\right] - \frac{80}{133} \mathrm{Li}_3 \left[-\frac{1}{8}\right] \,.\end{aligned} \tag{27}$$

Eq. (11)−Eq. (27) gives the BBP-ready identity

$$\begin{aligned}0 &= -\frac{88}{21} \mathrm{Li}_3 \left[\frac{1}{2}\right] - \frac{132}{133} \mathrm{Li}_3 \left[\frac{1}{4}\right] + \frac{88}{399} \mathrm{Li}_3 \left[-\frac{1}{8}\right] - \frac{1584}{133} \operatorname{Re} \mathrm{Li}_3 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right)\right]\\
&\quad + \frac{704}{133} \operatorname{Re} \mathrm{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] + \frac{704}{399} \operatorname{Re} \mathrm{Li}_3 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] \,,\end{aligned} \tag{28}$$

which yields the zero relation

$$
\begin{aligned}
0 &= P(3, 2^{12}, 24, (2^{11}, -19\cdot 2^{11}, 5\cdot 2^{14}, -2^{11}, -2^{9}, -23\cdot 2^{10}, 256, -27\cdot 2^{10},\\
&\quad -5\cdot 2^{11}, -19\cdot 2^{7}, -64, -7\cdot 2^{9}, -32, -19\cdot 2^{5}, -5\cdot 2^{8}, -27\cdot 2^{6}, 8,\\
&\quad -23\cdot 2^{4}, -4, -8, 160, -38, 1, 0))\,. \qquad (29)
\end{aligned}
$$

Eq. (29) was originally discovered by Bailey [2], using his PSLQ program.

5.2 Base 2^{60} Zero Relations

Solving Eqs. (2), (8) and (10) for $\zeta(3)$ and subtracting Eq. (21) gives the identity

$$
\begin{aligned}
0 &= \frac{50}{21}\,\mathrm{Li}_3\left[\frac{1}{2}\right] - \frac{5}{7}\,\mathrm{Li}_3\left[\frac{1}{4}\right] + \frac{10}{63}\,\mathrm{Li}_3\left[-\frac{1}{8}\right] - \frac{319}{63}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right]\\
&\quad -\frac{236}{63}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - \frac{8}{9}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\
&\quad +\frac{68}{63}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\,, \qquad (30)
\end{aligned}
$$

which leads to the zero relation

$$
\begin{aligned}
0 &= P(3, 2^{60}, 120, (7\cdot 2^{59}, -3\cdot 5^2\cdot 2^{60}, 1579\cdot 2^{57}, -29\cdot 2^{60}, -3\cdot 23\cdot 31\cdot 2^{56},\\
&\quad 3\cdot 5\cdot 2^{60}, 7\cdot 2^{56}, 67\cdot 2^{59}, -1579\cdot 2^{54}, -3\cdot 5^2\cdot 2^{56}, -7\cdot 2^{54}, -5\cdot 11\cdot 43\cdot 2^{53},\\
&\quad -7\cdot 2^{53}, -3\cdot 5^2\cdot 2^{54}, 3\cdot 7\cdot 13\cdot 2^{52}, 67\cdot 2^{55}, 7\cdot 2^{51}, 3\cdot 5\cdot 2^{54}, -7\cdot 2^{50},\\
&\quad 3\cdot 631\cdot 2^{49}, 1579\cdot 2^{48}, -3\cdot 5^2\cdot 2^{50}, 7\cdot 2^{48}, 5^3\cdot 17\cdot 2^{47}, 3\cdot 23\cdot 31\cdot 2^{46},\\
&\quad -3\cdot 5^2\cdot 2^{48}, 1579\cdot 2^{45}, -29\cdot 2^{48}, -7\cdot 2^{45}, 3\cdot 5\cdot 2^{48}, 7\cdot 2^{44}, 67\cdot 2^{47},\\
&\quad -1579\cdot 2^{42}, -3\cdot 5^2\cdot 2^{44}, -3\cdot 23\cdot 31\cdot 2^{41}, -5\cdot 11\cdot 43\cdot 2^{41}, -7\cdot 2^{41},\\
&\quad -3\cdot 5^2\cdot 2^{42}, -1579\cdot 2^{39}, -3^4\cdot 13\cdot 2^{39}, 7\cdot 2^{39}, 3\cdot 5\cdot 2^{42}, -7\cdot 2^{38},\\
&\quad -29\cdot 2^{40}, -3\cdot 7\cdot 13\cdot 2^{37}, -3\cdot 5^2\cdot 2^{38}, 7\cdot 2^{36}, 5^3\cdot 17\cdot 2^{35}, 7\cdot 2^{35},\\
&\quad -3\cdot 5^2\cdot 2^{36}, 1579\cdot 2^{33}, -29\cdot 2^{36}, -7\cdot 2^{33}, 3\cdot 5\cdot 2^{36}, 3\cdot 23\cdot 31\cdot 2^{31},\\
&\quad 67\cdot 2^{35}, -1579\cdot 2^{30}, -3\cdot 5^2\cdot 2^{32}, -7\cdot 2^{30}, -3\cdot 5\cdot 2^{33}, -7\cdot 2^{29},\\
&\quad -3\cdot 5^2\cdot 2^{30}, -1579\cdot 2^{27}, 67\cdot 2^{31}, 3\cdot 23\cdot 31\cdot 2^{26}, 3\cdot 5\cdot 2^{30}, -7\cdot 2^{26},\\
&\quad -29\cdot 2^{28}, 1579\cdot 2^{24}, -3\cdot 5^2\cdot 2^{26}, 7\cdot 2^{24}, 5^3\cdot 17\cdot 2^{23}, 7\cdot 2^{23},\\
&\quad -3\cdot 5^2\cdot 2^{24}, -3\cdot 7\cdot 13\cdot 2^{22}, -29\cdot 2^{24}, -7\cdot 2^{21}, 3\cdot 5\cdot 2^{24}, 7\cdot 2^{20},\\
&\quad -3^4\cdot 13\cdot 2^{19}, -1579\cdot 2^{18}, -3\cdot 5^2\cdot 2^{20}, -7\cdot 2^{18}, -5\cdot 11\cdot 43\cdot 2^{17},\\
&\quad -3\cdot 23\cdot 31\cdot 2^{16}, -3\cdot 5^2\cdot 2^{18}, -1579\cdot 2^{15}, 67\cdot 2^{19}, 7\cdot 2^{15}, 3\cdot 5\cdot 2^{18},\\
&\quad -7\cdot 2^{14}, -29\cdot 2^{16}, 1579\cdot 2^{12}, -3\cdot 5^2\cdot 2^{14}, 3\cdot 23\cdot 31\cdot 2^{11}, 5^3\cdot 17\cdot 2^{11},\\
&\quad 7\cdot 2^{11}, -3\cdot 5^2\cdot 2^{12}, 1579\cdot 2^{9}, 3\cdot 631\cdot 2^{9}, -7\cdot 2^{9}, 3\cdot 5\cdot 2^{12}, 7\cdot 2^{8},\\
&\quad 67\cdot 2^{11}, 3\cdot 7\cdot 13\cdot 2^{7}, -3\cdot 5^2\cdot 2^{8}, -7\cdot 2^{6}, -5\cdot 11\cdot 43\cdot 2^{5}, -7\cdot 2^{5},\\
&\quad -3\cdot 5^2\cdot 2^{6}, -1579\cdot 2^{3}, 67\cdot 2^{7}, 7\cdot 2^{3}, 3\cdot 5\cdot 2^{6}, -3\cdot 23\cdot 31\cdot 2,\\
&\quad -29\cdot 2^{4}, 1579, -3\cdot 5^2\cdot 2^{2}, 7, 0)) \qquad (31)
\end{aligned}
$$

Solving Eqs. (5), (8) and (10) for $\zeta(3)$ and subtracting Eq. (21) gives the identity

$$\begin{aligned} 0 \quad = \quad & -\frac{30}{119}\,\mathrm{Li}_3\left[\frac{1}{4}\right] + \frac{20}{357}\,\mathrm{Li}_3\left[-\frac{1}{8}\right] - \frac{7}{3}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] \\ & -\frac{52}{357}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + \frac{8}{357}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\ & +\frac{76}{357}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \end{aligned} \tag{32}$$

which produces the zero relation

$$\begin{aligned} 0 \quad = \quad & P(3, 2^{60}, 120, (2^{59}, 0, -353 \cdot 2^{57}, 7 \cdot 2^{62}, 3 \cdot 7 \cdot 113 \cdot 2^{56}, -3^3 \cdot 5 \cdot 2^{59}, 2^{56}, \\ & -97 \cdot 2^{60}, 353 \cdot 2^{54}, 0, -2^{54}, 5 \cdot 331 \cdot 2^{53}, -2^{53}, 0, -3 \cdot 337 \cdot 2^{52}, -97 \cdot 2^{56}, 2^{51}, \\ & -3^3 \cdot 5 \cdot 2^{53}, -2^{50}, -3^2 \cdot 239 \cdot 2^{49}, -353 \cdot 2^{48}, 0, 2^{48}, -5^3 \cdot 19 \cdot 2^{47}, -3 \cdot 7 \cdot 113 \cdot 2^{46}, \\ & 0, -353 \cdot 2^{45}, 7 \cdot 2^{50}, -2^{45}, -3^3 \cdot 5 \cdot 2^{47}, 2^{44}, -97 \cdot 2^{48}, 353 \cdot 2^{42}, 0, 3 \cdot 7 \cdot 113 \cdot 2^{41}, \\ & 5 \cdot 331 \cdot 2^{41}, -2^{41}, 0, 353 \cdot 2^{39}, -3^6 \cdot 2^{39}, 2^{39}, -3^3 \cdot 5 \cdot 2^{41}, -2^{38}, 7 \cdot 2^{42}, \\ & 3 \cdot 337 \cdot 2^{37}, 0, 2^{36}, -5^3 \cdot 19 \cdot 2^{35}, 2^{35}, 0, -353 \cdot 2^{33}, 7 \cdot 2^{38}, -2^{33}, -3^3 \cdot 5 \cdot 2^{35}, \\ & -3 \cdot 7 \cdot 113 \cdot 2^{31}, -97 \cdot 2^{36}, 353 \cdot 2^{30}, 0, -2^{30}, -3^2 \cdot 5 \cdot 2^{33}, -2^{29}, 0, 353 \cdot 2^{27}, \\ & -97 \cdot 2^{32}, -3 \cdot 7 \cdot 113 \cdot 2^{26}, -3^3 \cdot 5 \cdot 2^{29}, -2^{26}, 7 \cdot 2^{30}, -353 \cdot 2^{24}, 0, 2^{24}, \\ & -5^3 \cdot 19 \cdot 2^{23}, 2^{23}, 0, 3 \cdot 337 \cdot 2^{22}, 7 \cdot 2^{26}, -2^{21}, -3^3 \cdot 5 \cdot 2^{23}, 2^{20}, -3^6 \cdot 2^{19}, \\ & 353 \cdot 2^{18}, 0, -2^{18}, 5 \cdot 331 \cdot 2^{17}, 3 \cdot 7 \cdot 113 \cdot 2^{16}, 0, 353 \cdot 2^{15}, -97 \cdot 2^{20}, 2^{15}, \\ & -3^3 \cdot 5 \cdot 2^{17}, -2^{14}, 7 \cdot 2^{18}, -353 \cdot 2^{12}, 0, -3 \cdot 7 \cdot 113 \cdot 2^{11}, -5^3 \cdot 19 \cdot 2^{11}, 2^{11}, 0, \\ & -353 \cdot 2^{9}, -3^2 \cdot 239 \cdot 2^{9}, -2^{9}, -3^3 \cdot 5 \cdot 2^{11}, 2^{8}, -97 \cdot 2^{12}, -3 \cdot 337 \cdot 2^{7}, 0, -2^{6}, \\ & 5 \cdot 331 \cdot 2^{5}, -2^{5}, 0, 353 \cdot 2^{3}, -97 \cdot 2^{8}, 2^{3}, -3^3 \cdot 5 \cdot 2^{5}, \\ & 3 \cdot 7 \cdot 113 \cdot 2, 7 \cdot 2^{6}, -353, 0, 1, 0)) \end{aligned} \tag{33}$$

6 Conclusion

Using a clear and straightforward approach, we have obtained, for the first time in literature, proved binary BBP-type formulas for π^3 and $\pi \log^2 2$. We also discovered new BBP-type formulas for the remaining members of the trilogarithm family, namely $\zeta(3)$, $\pi^2 \log 2$ and $\log^3 2$. New binary degree 3 zero relations are also proved.

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