New Degree 4 Binary BBP-type Formulas and a Zero Relation*

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Abstract

We present new Binary BBP-type formulas for π^4, $\pi^2\log^2 2$, $\log^4 2$ and two linear combinations of $\mathrm{Cl}_4(\pi/2)$, $\pi\log^3 2$ and $\pi^3\log 2$. Known but hitherto unproved formulas are also now formally proved. Finally we obtain a base 2^{60}, length 120 degree 4 zero relation.

Contents

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1 Introduction

BBP-type formulas are formulas of the form

$$\alpha = \sum_{k=0}^{\infty} 1/b^k \sum_{j=1}^{l} a_j/(kl+j)^s$$

where s, b, l and a_j are integers, and α is some constant. Formulas of this type were first introduced in a 1996 paper [1], where a formula of this type for π was given. Such formulas allow digit extraction — the i-th digit of a mathematical constant α in base b can be calculated directly, without needing to compute any of the previous $i-1$ digits, by means of simple algorithms that do not require multiple-precision arithmetic [2].

Apart from digit extraction, another reason the study of BBP-type formulas has continued to attract attention is that BBP-type constants are conjectured to be either rational or normal to base b [3, 4, 5], that is their base-b digits are randomly distributed.

BBP-type formulas are usually discovered experimentally, through computer searches, by using Bailey and Ferguson's PSLQ (Partial Sum of Least Squares) algorithm [6] or its variations. PSLQ and other integer relation finding schemes typically do not suggest proofs [4, 7]. Formal proofs must be developed after the formulas have been discovered.

In this paper, we will derive explicit binary BBP-type formulas for π^4, $\pi^2 \log^2 2$, $\log^4 2$ and two linear combinations of $\mathrm{Cl}_4(\pi/2)$, $\pi \log^3 2$ and $\pi^3 \log 2$. We will also give formal proofs of the known but hitherto unproved formulas for these SC^* constants. To conclude we will obtain a degree 4 binary zero relation.

2 Generators of Degree 4 Binary BBP-type Formulas

The two-variable degree 4 polylogarithm functional equation (Eq. 7.90 pg. 211 of Lewin's book [8]) reads

$$\begin{aligned}
&\mathrm{Li}_4\left[-\frac{x^2y\eta}{\xi}\right] + \mathrm{Li}_4\left[-\frac{y^2x\xi}{\eta}\right] + \mathrm{Li}_4\left[\frac{x^2y}{\eta^2\xi}\right] + \mathrm{Li}_4\left[\frac{y^2x}{\xi^2\eta}\right] \\
&= 6\ \mathrm{Li}_4\left[xy\right] + 6\ \mathrm{Li}_4\left[\frac{xy}{\eta\,\xi}\right] + 6\ \mathrm{Li}_4\left[-\frac{xy}{\eta}\right] + 6\ \mathrm{Li}_4\left[-\frac{xy}{\xi}\right] \\
&+ 3\ \mathrm{Li}_4\left[x\eta\right] + 3\ \mathrm{Li}_4\left[y\xi\right] + 3\ \mathrm{Li}_4\left[\frac{x}{\eta}\right] + 3\ \mathrm{Li}_4\left[\frac{y}{\xi}\right] + 3\ \mathrm{Li}_4\left[-\frac{x\eta}{\xi}\right] \\
&+ 3\ \mathrm{Li}_4\left[-\frac{y\xi}{\eta}\right] + 3\ \mathrm{Li}_4\left[-\frac{x}{\eta\,\xi}\right] + 3\ \mathrm{Li}_4\left[-\frac{y}{\eta\,\xi}\right] - 6\ \mathrm{Li}_4\left[x\right] \\
&- 6\ \mathrm{Li}_4\left[y\right] - 6\ \mathrm{Li}_4\left[-\frac{x}{\xi}\right] - 6\ \mathrm{Li}_4\left[-\frac{y}{\eta}\right] + 3/2\ \log^2\xi\log^2\eta\,,
\end{aligned} \tag{1}$$

where $\xi = 1-x$, $\eta = 1-y$.

In the above equation, Li is the notation for the polylogarithm function defined by

$$\mathrm{Li}_s[z] = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad |z| \leq 1\,.$$

When $p = 1$ we have

$$\begin{aligned} \mathrm{Li}_{2n}[e^{ix}] &= \mathrm{Gl}_{2n}(x) + i\mathrm{Cl}_{2n}(x) \\ \mathrm{Li}_{2n+1}[e^{ix}] &= \mathrm{Cl}_{2n+1}(x) + i\mathrm{Gl}_{2n+1}(x)\,, \end{aligned} \tag{2}$$

where Gl and Cl are Clausen sums [8] defined, for $n \in \mathbb{Z}^+$ by

$$\begin{aligned} \mathrm{Cl}_{2n}(x) &= \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2n}}, \quad \mathrm{Cl}_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{\cos kx}{k^{2n+1}} \\ \mathrm{Gl}_{2n}(x) &= \sum_{k=1}^{\infty} \frac{\cos kx}{k^{2n}}, \quad \mathrm{Gl}_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2n+1}}\,. \end{aligned} \tag{3}$$

We shall find the following formulas useful:

$$\begin{aligned} \mathrm{Gl}_{2n}(x) &= (-1)^{1+[n/2]} 2^{n-1} \pi^n \mathrm{B}_n(x/2\pi)/n! \\ \frac{1}{m^{n-1}} \mathrm{Cl}_n(mx) &= \sum_{r=0}^{m-1} \mathrm{Cl}_n(x + 2\pi r/m)\,. \end{aligned} \tag{4}$$

Here $[n/2]$ denotes the integer part of $n/2$ and B_n are the Bernoulli polynomials defined by

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} \frac{\mathrm{B}_n(x) t^n}{n!}\,.$$

Evaluating Eq. (1) at coordinates $(1/2, 1/2)$ gives

$$\begin{aligned} &\frac{\pi^4}{9} - \frac{1}{2}\pi^2 \log^2 2 + \frac{5}{4} \log^4 2 \\ &\quad = 24\mathrm{Li}_4\left[\frac{1}{2}\right] + 2\mathrm{Li}_4\left[-\frac{1}{8}\right] - \frac{27}{2}\mathrm{Li}_4\left[\frac{1}{4}\right]\,. \end{aligned} \tag{5}$$

Evaluating Eq. (1) at coordinates (i, i), simplifying and taking real and imaginary parts, we obtain

$$\begin{aligned} &\frac{349\pi^4}{9216} - \frac{7\pi^2 \log^2 2}{128} + \frac{5}{64} \log^4 2 \\ &\quad = 2\mathrm{ReLi}_4\left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] + 12\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] \\ &\quad - \frac{27}{4}\mathrm{ReLi}_4\left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right)\right] - 6\mathrm{Li}_4\left[\frac{1}{2}\right] \end{aligned} \tag{6}$$

and

$$
\begin{aligned}
&20\mathrm{Cl}_4\left(\frac{\pi}{2}\right)+\frac{3\pi\log^3 2}{32}-\frac{27\pi^3\log 2}{128}\\
&\quad=-2\mathrm{ImLi}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]-\frac{27}{4}\mathrm{ImLi}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right]\\
&\quad\quad+36\mathrm{ImLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right].
\end{aligned}
\tag{7}
$$

Evaluating Eq. (1) at $(1+i,1/2)$, simplifying and taking the real part yields

$$
\begin{aligned}
&\frac{1697\pi^4}{9216}-\frac{137\pi^2\log^2 2}{384}+\frac{115}{192}\log^4 2\\
&\quad=-14\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right]+30\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\
&\quad-6\mathrm{ReLi}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{2}\right)\right]-7\mathrm{Li}_4\left[\frac{1}{2}\right]+\frac{11}{8}\mathrm{Li}_4\left[-\frac{1}{4}\right]
\end{aligned}
\tag{8}
$$

Finally, evaluating the identity at $((1-i)/2,1/2)$ and taking real and imaginary parts, we find

$$
\begin{aligned}
&\frac{1265\pi^4}{18432}-\frac{113\pi^2\log^2 2}{768}+\frac{91}{384}\log^4 2\\
&\quad=\mathrm{ReLi}_4\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{\pi i}{4}\right)\right]-12\mathrm{ReLi}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\
&\quad\quad+5\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]-7\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right]\\
&\quad\quad\quad-\frac{95}{8}\mathrm{ReLi}_4\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right]+\frac{11}{8}\mathrm{Li}_4\left[-\frac{1}{4}\right]+6\mathrm{Li}_4\left[\frac{1}{2}\right]
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
&12\mathrm{Cl}_4\left(\frac{\pi}{2}\right)+\frac{29\pi\log^3 2}{192}-\frac{47\pi^3\log 2}{256}\\
&\quad=-6\mathrm{ImLi}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]+\frac{95}{8}\mathrm{ImLi}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right]\\
&\quad\quad+\mathrm{ImLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]+7\mathrm{ImLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right]\\
&\quad\quad\quad-\mathrm{ImLi}_4\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right].
\end{aligned}
\tag{10}
$$

We are now ready to derive formulas for π^4, $\pi^2\log^2 2$, $\log^4 2$, and the two linear combinations of $\mathrm{Cl}_4(\pi/2)$, $\pi\log^3 2$ and $\pi^3\log 2$. In order to save space, we will give the formulas using the compact P-notation [2]:

$$P(s,b,l,A) \equiv \sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{l} \frac{a_j}{(kl+j)^s} \,,$$

where s, b and l are integers, and $A = (a_1, a_2, \ldots, a_n)$ is a vector of integers.

3 Base 2^{12}, Length 24 Binary BBP-type Formulas

Solving Eqs. (5), (6) and (8) simultaneously, we find

$$\begin{aligned}
\pi^4 = {} & \frac{27648}{41}\mathrm{ReLi}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + \frac{51840}{41}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
& + \frac{24192}{41}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] - \frac{58320}{41}\mathrm{ReLi}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] \\
& - \frac{32832}{41}\mathrm{Li}_4\left[\frac{1}{2}\right] - \frac{3888}{41}\mathrm{Li}_4\left[\frac{1}{4}\right] + \frac{576}{41}\mathrm{Li}_4\left[-\frac{1}{8}\right] - \frac{2376}{41}\mathrm{Li}_4\left[-\frac{1}{4}\right] \,,
\end{aligned} \tag{11}$$

$$\begin{aligned}
\pi^2\log^2 2 = {} & \frac{98944}{123}\mathrm{ReLi}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + \frac{47408}{41}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
& + \frac{31920}{41}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] - \frac{65142}{41}\mathrm{ReLi}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] \\
& - \frac{30200}{41}\mathrm{Li}_4\left[\frac{1}{2}\right] - \frac{6606}{41}\mathrm{Li}_4\left[\frac{1}{4}\right] + \frac{2936}{123}\mathrm{Li}_4\left[-\frac{1}{8}\right] - \frac{3135}{41}\mathrm{Li}_4\left[-\frac{1}{4}\right]
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\log^2 4 = {} & \frac{161024}{615}\mathrm{ReLi}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + \frac{71776}{205}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
& + \frac{53088}{205}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] - \frac{104364}{205}\mathrm{ReLi}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] \\
& - \frac{41872}{205}\mathrm{Li}_4\left[\frac{1}{2}\right] - \frac{13698}{205}\mathrm{Li}_4\left[\frac{1}{4}\right] + \frac{6088}{615}\mathrm{Li}_4\left[-\frac{1}{8}\right] - \frac{5214}{205}\mathrm{Li}_4\left[-\frac{1}{4}\right] \,.
\end{aligned} \tag{13}$$

Formulas (11), (12) and (13) will facilitate the derivation of base 2^{12}, length 24 BBP-type formulas for the respective polylogarithm constants.

3.1 Binary Formula for π^4

Using the identity

$$\mathrm{Re}\,\mathrm{Li}_4\left[pe^{iq}\right] = \sum_{k=1}^{\infty} \frac{p^k \cos(kq)}{k^4} \qquad p \in [0,1], q \in [0,2\pi] \,,$$

each term of Eq. (11) can be written as a BBP-type formula of length 24 and base 2^{12}. Doing this judiciously and combining according to Eq. (11), we find the base 2^{12}, length 24 binary BBP-type formula for π^4 to be:

$$\begin{aligned}\pi^4 \quad = \quad & \frac{1}{164}P(4, 2^{12}, 24, (27\cdot 2^{11}, -513\cdot 2^{11}, 135\cdot 2^{14}, -27\cdot 2^{11}, -27\cdot 2^{9}, \\ & -621\cdot 2^{10}, 27\cdot 2^{8}, -729\cdot 2^{10}, -135\cdot 2^{11}, -513\cdot 2^{7}, -27\cdot 2^{6}, \\ & -189\cdot 2^{9}, -27\cdot 2^{5}, -513\cdot 2^{5}, -135\cdot 2^{8}, -729\cdot 2^{6}, 216, -621\cdot 2^{4}, \\ & -108, -216, 135\cdot 2^{5}, -1026, 27, 0))\,. \end{aligned} \tag{14}$$

This proves formula 45 of the BBP Compendium, first obtained by D. H. Bailey, using his PSLQ program.

3.2 Binary Formula for $\pi^2 \log^2 2$

Writing each term of Eq. (12) as a BBP-type formula of length 24 and base 2^{12} and forming the indicated combination, we find the base 2^{12}, length 24 binary BBP-type formula for $\pi^2 \log^2 2$ to be:

$$\begin{aligned}\pi^2\log^2 2 \quad = \quad & \frac{1}{41\cdot 2^5}P(4, 2^{12}, 24, (121\cdot 2^{11}, -3775\cdot 2^{11}, 10375\cdot 2^{11}, -1597\cdot 2^{11}, \\ & -121\cdot 2^{9}, -3421\cdot 2^{11}, 121\cdot 2^{8}, -7695\cdot 2^{10}, -10375\cdot 2^{8}, -3775\cdot 2^{7}, \\ & -121\cdot 2^{6}, -3539\cdot 2^{8}, -121\cdot 2^{5}, -3775\cdot 2^{5}, -10375\cdot 2^{5}, -7695\cdot 2^{6}, \\ & 121\cdot 2^{3}, -3421\cdot 2^{5}, -484, -1597\cdot 2^{3}, 41500, -7550, 121, 0))\,. \end{aligned} \tag{15}$$

This proves formula 47 of the BBP Compendium, first obtained by D. H. Bailey, using his PSLQ program.

3.3 Binary Formula for $\log^4 2$

Using Eq. (13) and proceeding exactly as in the previous two sections, we find

$$\begin{aligned}\log^4 2 \quad = \quad & \frac{1}{205\cdot 2^5}P(4, 2^{12}, 24, (73\cdot 2^{12}, -2617\cdot 2^{12}, 8455\cdot 2^{12}, -2533\cdot 2^{12}, \\ & -73\cdot 2^{10}, -25781\cdot 2^{9}, 73\cdot 2^{9}, -6891\cdot 2^{11}, -8455\cdot 2^{9}, -2617\cdot 2^{8}, \\ & -73\cdot 2^{7}, -23551\cdot 2^{6}, -73\cdot 2^{6}, -2617\cdot 2^{6}, -8455\cdot 2^{6}, -6891\cdot 2^{7}, \\ & 73\cdot 2^{4}, -25781\cdot 2^{3}, -73\cdot 2^{3}, -2533\cdot 2^{4}, 8455\cdot 2^{3}, -10468, \\ & 146, -615))\,. \end{aligned} \tag{16}$$

This proves formula 46 of the BBP Compendium, obtained earlier by D. H. Bailey, using his PSLQ program.

3.4 Binary Formula for a Linear Combination of $\mathrm{Cl}_4(\pi/2)$, $\pi\log^3 2$ and $\pi^3\log 2$

Using the identity

$$\mathrm{Im}\,\mathrm{Li}_4\left[pe^{iq}\right] = \sum_{k=1}^{\infty}\frac{p^k\sin(kq)}{k^4} \qquad p\in[0,1], q\in[0,2\pi]\,,$$

each term of Eq. (7) can be written as a BBP-type formula of length 24 and base 2^{12}. Doing this judiciously and combining according to Eq. (7), we find:

$$\begin{aligned} 20\mathrm{Cl}_4\left(\frac{\pi}{2}\right)+\frac{3\pi\log^3 2}{32}-\frac{27\pi^3\log 2}{128} \quad = \quad & \frac{9}{2^{10}}P(4,2^{12},24,(2^{11},-2^{12},-7\cdot 2^9,0,-2^9,\\ & -5\cdot 2^8,-2^8,0,-7\cdot 2^6,-2^8,2^6,0,-2^5,2^6,7\cdot 2^3,0,\\ & 2^3,5\cdot 2^2,2^2,0,7,2^2,-1,0))\,. \end{aligned} \tag{17}$$

4 Base 2^{60}, length 120 Binary BBP-type Formulas

Solving Eqs. (6), (8) and (9) simultaneously, we find

$$\begin{aligned} \pi^4 = & \frac{57600}{71}\mathrm{ReLi}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]+\frac{442368}{497}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\ & +\frac{13824}{71}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right]+\frac{239328}{497}\mathrm{ReLi}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right]\\ & -\frac{34560}{497}\mathrm{ReLi}_4\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]-\frac{432000}{497}\mathrm{Li}_4\left[\frac{1}{2}\right]-\frac{4752}{71}\mathrm{Li}_4\left[-\frac{1}{4}\right]\,, \end{aligned} \tag{18}$$

$$\begin{aligned} \pi^2\log^2 2 = & \frac{73632}{71}\mathrm{ReLi}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]+\frac{258592}{497}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\ & +\frac{7584}{71}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right]+\frac{818152}{497}\mathrm{ReLi}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right]\\ & -\frac{58720}{497}\mathrm{ReLi}_4\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]-\frac{423872}{497}\mathrm{Li}_4\left[\frac{1}{2}\right]-\frac{6512}{71}\mathrm{Li}_4\left[-\frac{1}{4}\right] \end{aligned} \tag{19}$$

and

$$\begin{aligned} \log^4 2 = & \frac{25440}{71}\mathrm{ReLi}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]+\frac{42928}{497}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\ & -\frac{1392}{71}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right]+\frac{413758}{497}\mathrm{ReLi}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right]\\ & -\frac{24352}{497}\mathrm{ReLi}_4\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]-\frac{125480}{497}\mathrm{Li}_4\left[\frac{1}{2}\right]-\frac{2255}{71}\mathrm{Li}_4\left[-\frac{1}{4}\right]\,. \end{aligned} \tag{20}$$

4.1 Binary Formula for π^4

As usual, the identity

$$\mathrm{Re}\,\mathrm{Li}_4\left[pe^{iq}\right] = \sum_{k=1}^{\infty} \frac{p^k \cos(kq)}{k^4} \quad p \in [0,1], q \in [0, 2\pi]\,,$$

allows one to write each term of Eq. (18) as a BBP-type formula of length 120 and base 2^{60}. Doing this judiciously and combining according to Eq. (18), we find the base 2^{60}, length 120 binary BBP-type formula for π^4 to be:

$$\begin{aligned}
\pi^4 \quad = \quad & \frac{675}{7\cdot 71\cdot 2^{51}} P(4, 2^{60}, 120, (2^{59}, -5\cdot 2^{61}, 11\cdot 17\cdot 2^{57}, -2^{61}, -127\cdot 2^{56}, \\
& -5\cdot 2^{59}, 2^{56}, -2^{61}, -11\cdot 17\cdot 2^{54}, -5\cdot 2^{57}, -2^{54}, -5\cdot 41\cdot 2^{53}, \\
& -2^{53}, -5\cdot 2^{55}, -31\cdot 2^{52}, -2^{57}, 2^{51}, -5\cdot 2^{53}, -2^{50}, 109\cdot 2^{49}, \\
& 11\cdot 17\cdot 2^{48}, -5\cdot 2^{51}, 2^{48}, 5^3\cdot 2^{47}, 127\cdot 2^{46}, -5\cdot 2^{49}, 11\cdot 17\cdot 2^{45}, \\
& -2^{49}, -2^{45}, -5\cdot 2^{47}, 2^{44}, -2^{49}, -11\cdot 17\cdot 2^{42}, -5\cdot 2^{45}, -127\cdot 2^{41}, \\
& -5\cdot 41\cdot 2^{41}, -2^{41}, -5\cdot 2^{43}, -11\cdot 17\cdot 2^{39}, -3^3\cdot 7\cdot 2^{39}, 2^{39}, -5\cdot 2^{41}, \\
& -2^{38}, -2^{41}, 31\cdot 2^{37}, -5\cdot 2^{39}, 2^{36}, 5^3\cdot 2^{35}, 2^{35}, -5\cdot 2^{37}, \\
& 11\cdot 17\cdot 2^{33}, -2^{37}, -2^{33}, -5\cdot 2^{35}, 127\cdot 2^{31}, -2^{37}, -11\cdot 17\cdot 2^{30}, \\
& -5\cdot 2^{33}, -2^{30}, -5\cdot 2^{33}, -2^{29}, -5\cdot 2^{31}, -11\cdot 17\cdot 2^{27}, -2^{33}, \\
& 127\cdot 2^{26}, -5\cdot 2^{29}, -2^{26}, -2^{29}, 11\cdot 17\cdot 2^{24}, -5\cdot 2^{27}, 2^{24}, \\
& 5^3\cdot 2^{23}, 2^{23}, -5\cdot 2^{25}, 31\cdot 2^{22}, -2^{25}, -2^{21}, -5\cdot 2^{23}, 2^{20}, \\
& -3^3\cdot 7\cdot 2^{19}, -11\cdot 17\cdot 2^{18}, -5\cdot 2^{21}, -2^{18}, -5\cdot 41\cdot 2^{17}, -127\cdot 2^{16}, \\
& -5\cdot 2^{19}, -11\cdot 17\cdot 2^{15}, -2^{21}, 2^{15}, -5\cdot 2^{17}, -2^{14}, -2^{17}, 11\cdot 17\cdot 2^{12}, \\
& -5\cdot 2^{15}, 127\cdot 2^{11}, 5^3\cdot 2^{11}, 2^{11}, -5\cdot 2^{13}, 11\cdot 17\cdot 2^{9}, 109\cdot 2^{9}, -2^{9}, \\
& -5\cdot 2^{11}, 2^{8}, -2^{13}, -31\cdot 2^{7}, -5\cdot 2^{9}, -2^{6}, -5\cdot 41\cdot 2^{5}, -2^{5}, -5\cdot 2^{7}, \\
& -11\cdot 17\cdot 2^{3}, -2^{9}, 2^{3}, -5\cdot 2^{5}, -127\cdot 2, -2^{5}, 17\cdot 11, -5\cdot 2^{3}, 1, 0))\,. \qquad (21)
\end{aligned}$$

4.2 Binary Formula for $\pi^2 \log^2 2$

Using Eq. (19) and following the same procedure as in the previous cases, the length 120 base 2^{60} binary BBP-type formula for $\pi^2 \log^2 2$ is obtained as

$$
\begin{aligned}
\pi^2 \log^2 2 \quad = \quad & \frac{1}{7 \cdot 71 \cdot 2^{54}} P(4, 2^{60}, 120, (13^2 \cdot 19 \cdot 2^{59}, -37 \cdot 179 \cdot 2^{63}, 5 \cdot 13 \cdot 19973 \cdot 2^{57}, \\
& -5 \cdot 1663 \cdot 2^{62}, -17 \cdot 179 \cdot 379 \cdot 2^{56}, -37 \cdot 179 \cdot 2^{61}, 13^2 \cdot 19 \cdot 2^{56}, -4931 \cdot 2^{60}, \\
& -5 \cdot 13 \cdot 19973 \cdot 2^{54}, -37 \cdot 179 \cdot 2^{59}, -13^2 \cdot 19 \cdot 2^{54}, -43 \cdot 36529 \cdot 2^{53}, -13^2 \cdot 19 \cdot 2^{53}, \\
& -37 \cdot 179 \cdot 2^{57}, -5 \cdot 15137 \cdot 2^{52}, -4931 \cdot 2^{56}, 13^2 \cdot 19 \cdot 2^{51}, -37 \cdot 179 \cdot 2^{55}, \\
& -13^2 \cdot 19 \cdot 2^{50}, 5 \cdot 176159 \cdot 2^{49}, 5 \cdot 13 \cdot 19973 \cdot 2^{48}, -37 \cdot 179 \cdot 2^{53}, 13^2 \cdot 19 \cdot 2^{48}, \\
& 5^5 \cdot 367 \cdot 2^{47}, 17 \cdot 179 \cdot 379 \cdot 2^{46}, -37 \cdot 179 \cdot 2^{51}, 5 \cdot 13 \cdot 19973 \cdot 2^{45}, -5 \cdot 1663 \cdot 2^{50}, \\
& -13^2 \cdot 19 \cdot 2^{45}, -37 \cdot 179 \cdot 2^{49}, 13^2 \cdot 19 \cdot 2^{44}, -4931 \cdot 2^{48}, -5 \cdot 13 \cdot 19973 \cdot 2^{42}, \\
& -37 \cdot 179 \cdot 2^{47}, -17 \cdot 179 \cdot 379 \cdot 2^{41}, -43 \cdot 36529 \cdot 2^{41}, -13^2 \cdot 19 \cdot 2^{41}, -37 \cdot 179 \cdot 2^{45}, \\
& -5 \cdot 13 \cdot 19973 \cdot 2^{39}, -3^5 \cdot 7 \cdot 13 \cdot 59 \cdot 2^{39}, 13^2 \cdot 19 \cdot 2^{39}, -37 \cdot 179 \cdot 2^{43}, -13^2 \cdot 19 \cdot 2^{38}, \\
& -5 \cdot 1663 \cdot 2^{42}, 5 \cdot 15137 \cdot 2^{37}, -37 \cdot 179 \cdot 2^{41}, 13^2 \cdot 19 \cdot 2^{36}, 5^5 \cdot 367 \cdot 2^{35}, \\
& 13^2 \cdot 19 \cdot 2^{35}, -37 \cdot 179 \cdot 2^{39}, 5 \cdot 13 \cdot 19973 \cdot 2^{33}, -5 \cdot 1663 \cdot 2^{38}, -13^2 \cdot 19 \cdot 2^{33}, \\
& -37 \cdot 179 \cdot 2^{37}, 17 \cdot 179 \cdot 379 \cdot 2^{31}, -4931 \cdot 2^{36}, -5 \cdot 13 \cdot 19973 \cdot 2^{30}, -37 \cdot 179 \cdot 2^{35}, \\
& -13^2 \cdot 19 \cdot 2^{30}, -37 \cdot 179 \cdot 2^{35}, -13^2 \cdot 19 \cdot 2^{29}, -37 \cdot 179 \cdot 2^{33}, -5 \cdot 13 \cdot 19973 \cdot 2^{27}, \\
& -4931 \cdot 2^{32}, 17 \cdot 179 \cdot 379 \cdot 2^{26}, -37 \cdot 179 \cdot 2^{31}, -13^2 \cdot 19 \cdot 2^{26}, -5 \cdot 1663 \cdot 2^{30}, \\
& 5 \cdot 13 \cdot 19973 \cdot 2^{24}, -37 \cdot 179 \cdot 2^{29}, 13^2 \cdot 19 \cdot 2^{24}, 5^5 \cdot 367 \cdot 2^{23}, 13^2 \cdot 19 \cdot 2^{23}, \\
& -37 \cdot 179 \cdot 2^{27}, 5 \cdot 15137 \cdot 2^{22}, -5 \cdot 1663 \cdot 2^{26}, -13^2 \cdot 19 \cdot 2^{21}, -37 \cdot 179 \cdot 2^{25}, \\
& 13^2 \cdot 19 \cdot 2^{20}, -3^5 \cdot 7 \cdot 13 \cdot 59 \cdot 2^{19}, -5 \cdot 13 \cdot 19973 \cdot 2^{18}, -37 \cdot 179 \cdot 2^{23}, \\
& -13^2 \cdot 19 \cdot 2^{18}, -43 \cdot 36529 \cdot 2^{17}, -17 \cdot 179 \cdot 379 \cdot 2^{16}, -37 \cdot 179 \cdot 2^{21}, -5 \cdot 13 \cdot 19973 \cdot 2^{15}, \\
& -4931 \cdot 2^{20}, 13^2 \cdot 19 \cdot 2^{15}, -37 \cdot 179 \cdot 2^{19}, -13^2 \cdot 19 \cdot 2^{14}, -5 \cdot 1663 \cdot 2^{18}, \\
& 5 \cdot 13 \cdot 19973 \cdot 2^{12}, -37 \cdot 179 \cdot 2^{17}, 17 \cdot 179 \cdot 379 \cdot 2^{11}, 5^5 \cdot 367 \cdot 2^{11}, 13^2 \cdot 19 \cdot 2^{11}, \\
& -37 \cdot 179 \cdot 2^{15}, 5 \cdot 13 \cdot 19973 \cdot 2^{9}, 5 \cdot 176159 \cdot 2^{9}, -13^2 \cdot 19 \cdot 2^{9}, -37 \cdot 179 \cdot 2^{13}, \\
& 13^2 \cdot 19 \cdot 2^{8}, -4931 \cdot 2^{12}, -5 \cdot 15137 \cdot 2^{7}, -37 \cdot 179 \cdot 2^{11}, -13^2 \cdot 19 \cdot 2^{6}, \\
& -43 \cdot 36529 \cdot 2^{5}, -13^2 \cdot 19 \cdot 2^{5}, -37 \cdot 179 \cdot 2^{9}, -5 \cdot 13 \cdot 19973 \cdot 2^{3}, -4931 \cdot 2^{8}, \\
& 13^2 \cdot 19 \cdot 2^{3}, -37 \cdot 179 \cdot 2^{7}, -17 \cdot 179 \cdot 379 \cdot 2, -5 \cdot 1663 \cdot 2^{6}, 13 \cdot 19973 \cdot 5, \\
& -37 \cdot 179 \cdot 2^{5}, 19 \cdot 13^2, 0)) \,. \qquad (22)
\end{aligned}
$$

4.3 Binary Formula for $\log^4 2$

Writing each component of the right hand side Eq. (20) as a length 120, base 2^{60} binary BBP-type formula, and forming the required combination, we obtain the base 2^{60}, length 120 binary BBP-type formula for $\log^4 2$ as

$$
\begin{aligned}
\log^4 2 \quad = \quad & \frac{1}{7 \cdot 71 \cdot 2^{54}} P(4, 2^{60}, 120, (823 \cdot 2^{59}, -5 \cdot 3137 \cdot 2^{60}, 11 \cdot 40829 \cdot 2^{57}, -18047 \cdot 2^{60}, \\
& -277 \cdot 1723 \cdot 2^{56}, -5 \cdot 3137 \cdot 2^{58}, 823 \cdot 2^{56}, 1181 \cdot 2^{59}, -11 \cdot 40829 \cdot 2^{54}, \\
& -5 \cdot 3137 \cdot 2^{56}, -823 \cdot 2^{54}, -595141 \cdot 2^{53}, -823 \cdot 2^{53}, -5 \cdot 3137 \cdot 2^{54}, \\
& 29 \cdot 457 \cdot 2^{52}, 1181 \cdot 2^{55}, 823 \cdot 2^{51}, -5 \cdot 3137 \cdot 2^{52}, -823 \cdot 2^{50}, \\
& 331249 \cdot 2^{49}, 11 \cdot 40829 \cdot 2^{48}, -5 \cdot 3137 \cdot 2^{50}, 823 \cdot 2^{48}, 19^2 \cdot 1301 \cdot 2^{47}, \\
& 277 \cdot 1723 \cdot 2^{46}, -5 \cdot 3137 \cdot 2^{48}, 11 \cdot 40829 \cdot 2^{45}, -18047 \cdot 2^{48}, -823 \cdot 2^{45}, \\
& -5 \cdot 3137 \cdot 2^{46}, 823 \cdot 2^{44}, 1181 \cdot 2^{47}, -11 \cdot 40829 \cdot 2^{42}, -5 \cdot 3137 \cdot 2^{44}, \\
& -277 \cdot 1723 \cdot 2^{41}, -595141 \cdot 2^{41}, -823 \cdot 2^{41}, -5 \cdot 3137 \cdot 2^{42}, -11 \cdot 40829 \cdot 2^{39}, \\
& -3 \cdot 7^2 \cdot 13 \cdot 239 \cdot 2^{39}, 823 \cdot 2^{39}, -5 \cdot 3137 \cdot 2^{40}, -823 \cdot 2^{38}, -18047 \cdot 2^{40}, \\
& -29 \cdot 457 \cdot 2^{37}, -5 \cdot 3137 \cdot 2^{38}, 823 \cdot 2^{36}, 19^2 \cdot 1301 \cdot 2^{35}, 823 \cdot 2^{35}, \\
& -5 \cdot 3137 \cdot 2^{36}, 11 \cdot 40829 \cdot 2^{33}, -18047 \cdot 2^{36}, -823 \cdot 2^{33}, -5 \cdot 3137 \cdot 2^{34}, \\
& 277 \cdot 1723 \cdot 2^{31}, 1181 \cdot 2^{35}, -11 \cdot 40829 \cdot 2^{30}, -5 \cdot 3137 \cdot 2^{32}, -823 \cdot 2^{30}, \\
& -29879 \cdot 2^{31}, -823 \cdot 2^{29}, -5 \cdot 3137 \cdot 2^{30}, -11 \cdot 40829 \cdot 2^{27}, 1181 \cdot 2^{31}, \\
& 277 \cdot 1723 \cdot 2^{26}, -5 \cdot 3137 \cdot 2^{28}, -823 \cdot 2^{26}, -18047 \cdot 2^{28}, 11 \cdot 40829 \cdot 2^{24}, \\
& -5 \cdot 3137 \cdot 2^{26}, 823 \cdot 2^{24}, 19^2 \cdot 1301 \cdot 2^{23}, 823 \cdot 2^{23}, -5 \cdot 3137 \cdot 2^{24}, \\
& -29 \cdot 457 \cdot 2^{22}, -18047 \cdot 2^{24}, -823 \cdot 2^{21}, -5 \cdot 3137 \cdot 2^{22}, 823 \cdot 2^{20}, \\
& -3 \cdot 7^2 \cdot 13 \cdot 239 \cdot 2^{19}, -11 \cdot 40829 \cdot 2^{18}, -5 \cdot 3137 \cdot 2^{20}, -823 \cdot 2^{18}, \\
& -595141 \cdot 2^{17}, -277 \cdot 1723 \cdot 2^{16}, -5 \cdot 3137 \cdot 2^{18}, -11 \cdot 40829 \cdot 2^{15}, \\
& 1181 \cdot 2^{19}, 823 \cdot 2^{15}, -5 \cdot 3137 \cdot 2^{16}, -823 \cdot 2^{14}, -18047 \cdot 2^{16}, \\
& 11 \cdot 40829 \cdot 2^{12}, -5 \cdot 3137 \cdot 2^{14}, 277 \cdot 1723 \cdot 2^{11}, 19^2 \cdot 1301 \cdot 2^{11}, \\
& 823 \cdot 2^{11}, -5 \cdot 3137 \cdot 2^{12}, 11 \cdot 40829 \cdot 2^{9}, 331249 \cdot 2^{9}, -823 \cdot 2^{9}, \\
& -5 \cdot 3137 \cdot 2^{10}, 823 \cdot 2^{8}, 1181 \cdot 2^{11}, 29 \cdot 457 \cdot 2^{7}, -5 \cdot 3137 \cdot 2^{8}, \\
& -823 \cdot 2^{6}, -595141 \cdot 2^{5}, -823 \cdot 2^{5}, -5 \cdot 3137 \cdot 2^{6}, -11 \cdot 40829 \cdot 2^{3}, \\
& 1181 \cdot 2^{7}, 823 \cdot 2^{3}, -5 \cdot 3137 \cdot 2^{4}, -277 \cdot 1723 \cdot 2, -18047 \cdot 2^{4}, \\
& 40829 \cdot 11, -5 \cdot 3137 \cdot 2^{2}, 823, -3 \cdot 7 \cdot 71 \cdot 2)) \, . \qquad (23)
\end{aligned}
$$

4.4 Binary Formula for a Linear Combination of $\mathrm{Cl}_4(\pi/2)$, $\pi \log^3 2$ and $\pi^3 \log 2$

As usual, using the identity

$$
\mathrm{Im}\ \mathrm{Li}_4 \left[p e^{iq} \right] = \sum_{k=1}^{\infty} \frac{p^k \sin(kq)}{k^4} \quad p \in [0,1], q \in [0, 2\pi] \, ,
$$

to write each polylogarithm term of Eq. (10) as a BBP-type formula of length 120 and base 2^{60} and forming the indicated linear combination we obtain

$$
\begin{aligned}
&12\mathrm{Cl}_4\left(\frac{\pi}{2}\right)+\frac{29\pi\log^3 2}{192}-\frac{47\pi^3\log 2}{256}\\
&=\frac{1}{2^{58}}P(4,2^{60},120,(2^{60},23\cdot 2^{60},\\
&-239\cdot 2^{57},0,-3\cdot 211\cdot 2^{55},-5\cdot 67\cdot 2^{56},-2^{57},0,-239\cdot 2^{54},\\
&-3^2\cdot 7^2\cdot 2^{53},2^{55},0,-2^{54},-23\cdot 2^{54},-3\cdot 7^2\cdot 2^{50},0,\\
&2^{52},5\cdot 67\cdot 2^{50},2^{51},0,239\cdot 2^{48},-23\cdot 2^{50},-2^{49},0,\\
&3\cdot 211\cdot 2^{45},23\cdot 2^{48},-239\cdot 2^{45},0,-2^{46},-3^2\cdot 5\cdot 2^{43},\\
&-2^{45},0,-239\cdot 2^{42},23\cdot 2^{44},3\cdot 211\cdot 2^{40},0,-2^{42},-23\cdot 2^{42},239\cdot 2^{39},\\
&0,2^{40},5\cdot 67\cdot 2^{38},2^{39},0,-3\cdot 7^2\cdot 2^{35},-23\cdot 2^{38},-2^{37},0,2^{36},\\
&-3^2\cdot 7^2\cdot 2^{33},-239\cdot 2^{33},0,-2^{34},-5\cdot 67\cdot 2^{32},-3\cdot 211\cdot 2^{30},0,\\
&-239\cdot 2^{30},23\cdot 2^{32},2^{31},0,-2^{30},-23\cdot 2^{30},239\cdot 2^{27},0,3\cdot 211\cdot 2^{25},\\
&5\cdot 67\cdot 2^{26},2^{27},0,239\cdot 2^{24},3^2\cdot 7^2\cdot 2^{23},-2^{25},0,2^{24},23\cdot 2^{24},\\
&3\cdot 7^2\cdot 2^{20},0,-2^{22},-5\cdot 67\cdot 2^{20},-2^{21},0,-239\cdot 2^{18},23\cdot 2^{20},2^{19},0,\\
&-3\cdot 211\cdot 2^{15},-23\cdot 2^{18},239\cdot 2^{15},0,2^{16},3^2\cdot 5\cdot 2^{13},2^{15},0,239\cdot 2^{12},\\
&-23\cdot 2^{14},-3\cdot 211\cdot 2^{10},0,2^{12},23\cdot 2^{12},-239\cdot 2^{9},0,-2^{10},\\
&-5\cdot 67\cdot 2^{8},-2^{9},0,3\cdot 7^2\cdot 2^{5},23\cdot 2^{8},2^{7},0,-2^{6},3^2\cdot 7^2\cdot 2^{3},\\
&239\cdot 2^{3},0,2^{4},5\cdot 67\cdot 2^{2},211\cdot 3,0,239,-23\cdot 2^{2},-2,0))\,. \qquad (24)
\end{aligned}
$$

5 Degree 4 Binary Zero Relation

Subtracting Eq. (18) from Eq. (11), we obtain the identity

$$
\begin{aligned}
0 = &-\frac{398592}{2911}\mathrm{ReLi}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]+\frac{7627392}{20377}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\
&+\frac{1150848}{2911}\mathrm{ReLi}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right]+\frac{34560}{497}\mathrm{ReLi}_4\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\
&-\frac{38797488}{20377}\mathrm{ReLi}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right]+\frac{1394496}{20377}\mathrm{Li}_4\left[\frac{1}{2}\right]\\
&-\frac{3888}{41}\mathrm{Li}_4\left[\frac{1}{4}\right]+\frac{576}{41}\mathrm{Li}_4\left[-\frac{1}{8}\right]+\frac{26136}{2911}\mathrm{Li}_4\left[-\frac{1}{4}\right]\,.
\end{aligned} \qquad (25)
$$

As usual, writing each component of Eq. (25) as a base 2^{60}, length 120 binary BBP-type formula and forming the indicated combination, we obtain the degree 4 binary zero relation:

$$
\begin{aligned}
0 \quad = \quad & P(4, 2^{60}, 120, (-31 \cdot 2^{59}, 3 \cdot 269 \cdot 2^{60}, -5 \cdot 61 \cdot 107 \cdot 2^{57}, \\
& 1553 \cdot 2^{60}, 3^2 \cdot 14243 \cdot 2^{56}, -3 \cdot 1051 \cdot 2^{60}, -31 \cdot 2^{56}, \\
& -9319 \cdot 2^{59}, 5 \cdot 61 \cdot 107 \cdot 2^{54}, 3 \cdot 269 \cdot 2^{56}, 31 \cdot 2^{54}, \\
& 31 \cdot 3187 \cdot 2^{53}, 31 \cdot 2^{53}, 3 \cdot 269 \cdot 2^{54}, -3^2 \cdot 5 \cdot 1061 \cdot 2^{52}, \\
& -9319 \cdot 2^{55}, -31 \cdot 2^{51}, -3 \cdot 1051 \cdot 2^{54}, 31 \cdot 2^{50}, -3 \cdot 38567 \cdot 2^{49}, \\
& -5 \cdot 61 \cdot 107 \cdot 2^{48}, 3 \cdot 269 \cdot 2^{50}, -31 \cdot 2^{48}, -5^5 \cdot 41 \cdot 2^{47}, \\
& -3^2 \cdot 14243 \cdot 2^{46}, 3 \cdot 269 \cdot 2^{48}, -5 \cdot 61 \cdot 107 \cdot 2^{45}, 1553 \cdot 2^{48}, \\
& 31 \cdot 2^{45}, -3 \cdot 1051 \cdot 2^{48}, -31 \cdot 2^{44}, -9319 \cdot 2^{47}, 5 \cdot 61 \cdot 107 \cdot 2^{42}, \\
& 3 \cdot 269 \cdot 2^{44}, 3^2 \cdot 14243 \cdot 2^{41}, 31 \cdot 3187 \cdot 2^{41}, 31 \cdot 2^{41}, \\
& 3 \cdot 269 \cdot 2^{42}, 5 \cdot 61 \cdot 107 \cdot 2^{39}, -3^4 \cdot 7 \cdot 37 \cdot 2^{39}, -31 \cdot 2^{39}, \\
& -3 \cdot 1051 \cdot 2^{42}, 31 \cdot 2^{38}, 1553 \cdot 2^{40}, 3^2 \cdot 5 \cdot 1061 \cdot 2^{37}, \\
& 3 \cdot 269 \cdot 2^{38}, -31 \cdot 2^{36}, -5^5 \cdot 41 \cdot 2^{35}, -31 \cdot 2^{35}, \\
& 3 \cdot 269 \cdot 2^{36}, -5 \cdot 61 \cdot 107 \cdot 2^{33}, 1553 \cdot 2^{36}, 31 \cdot 2^{33}, \\
& -3 \cdot 1051 \cdot 2^{36}, -3^2 \cdot 14243 \cdot 2^{31}, -9319 \cdot 2^{35}, \\
& 5 \cdot 61 \cdot 107 \cdot 2^{30}, 3 \cdot 269 \cdot 2^{32}, 31 \cdot 2^{30}, -3 \cdot 13 \cdot 47 \cdot 2^{33}, \\
& 31 \cdot 2^{29}, 3 \cdot 269 \cdot 2^{30}, 5 \cdot 61 \cdot 107 \cdot 2^{27}, -9319 \cdot 2^{31}, \\
& -3^2 \cdot 14243 \cdot 2^{26}, -3 \cdot 1051 \cdot 2^{30}, 31 \cdot 2^{26}, 1553 \cdot 2^{28}, \\
& -5 \cdot 61 \cdot 107 \cdot 2^{24}, 3 \cdot 269 \cdot 2^{26}, -31 \cdot 2^{24}, -5^5 \cdot 41 \cdot 2^{23}, \\
& -31 \cdot 2^{23}, 3 \cdot 269 \cdot 2^{24}, 3^2 \cdot 5 \cdot 1061 \cdot 2^{22}, 1553 \cdot 2^{24}, \\
& 31 \cdot 2^{21}, -3 \cdot 1051 \cdot 2^{24}, -31 \cdot 2^{20}, -3^4 \cdot 7 \cdot 37 \cdot 2^{19}, \\
& 5 \cdot 61 \cdot 107 \cdot 2^{18}, 3 \cdot 269 \cdot 2^{20}, 31 \cdot 2^{18}, 31 \cdot 3187 \cdot 2^{17}, \\
& 3^2 \cdot 14243 \cdot 2^{16}, 3 \cdot 269 \cdot 2^{18}, 5 \cdot 61 \cdot 107 \cdot 2^{15}, -9319 \cdot 2^{19}, \\
& -31 \cdot 2^{15}, -3 \cdot 1051 \cdot 2^{18}, 31 \cdot 2^{14}, 1553 \cdot 2^{16}, \\
& -5 \cdot 61 \cdot 107 \cdot 2^{12}, 3 \cdot 269 \cdot 2^{14}, -3^2 \cdot 14243 \cdot 2^{11}, \\
& -5^5 \cdot 41 \cdot 2^{11}, -31 \cdot 2^{11}, 3 \cdot 269 \cdot 2^{12}, \\
& -5 \cdot 61 \cdot 107 \cdot 2^9, -3 \cdot 38567 \cdot 2^9, 31 \cdot 2^9, -3 \cdot 1051 \cdot 2^{12}, \\
& -31 \cdot 2^8, -9319 \cdot 2^{11}, -3^2 \cdot 5 \cdot 1061 \cdot 2^7, 3 \cdot 269 \cdot 2^8, \\
& 31 \cdot 2^6, 31 \cdot 3187 \cdot 2^5, 31 \cdot 2^5, 3 \cdot 269 \cdot 2^6, \\
& 5 \cdot 61 \cdot 107 \cdot 2^3, -9319 \cdot 2^7, -31 \cdot 2^3, -3 \cdot 1051 \cdot 2^6, \\
& 3^2 \cdot 14243 \cdot 2, 1553 \cdot 2^4, -61 \cdot 107 \cdot 5, 3 \cdot 269 \cdot 2^2, -31, 0))\,. \qquad (26)
\end{aligned}
$$

6 Conclusion

In this paper, we have derived explicit binary BBP-type formulas for π^4, $\pi^2 \log^2 2$, $\log^4 2$ and two linear combinations of $\mathrm{Cl}_4(\pi/2)$, $\pi \log^3 2$ and $\pi^3 \log 2$. We also gave formal proofs of the known but hitherto unproved formulas for three of these SC^*

constants. Finally, we obtained a degree 4 binary zero relation, which according to Bailey ([2], Table I) is the only one.

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