Formal Proofs of Degree 5 Binary BBP-type Formulas

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Abstract

We give formal proofs for the binary BBP-type formulas for $\zeta(5)$, $\pi^4 \ln 2$, $\pi^2 \ln^3 2$ and $\ln^5 2$.

1 Proofs

Evaluating the two-variable degree 5 polylogarithm functional equation (Eq. 63 p5 of [1]) at coordinates $(1/2, 1/2)$ gives

$$\begin{aligned} &\frac{403}{4}\,\zeta(5) - \frac{2}{3}\,\pi^4 \ln 2 + \pi^2 \ln^3 2 - \frac{3}{2}\,\ln^5 2 \\ &\quad = 144\,\mathrm{Li}_5\left[\frac{1}{2}\right] - \frac{81}{2}\,\mathrm{Li}_5\left[\frac{1}{4}\right] + 4\,\mathrm{Li}_5\left[-\frac{1}{8}\right]. \end{aligned} \tag{1}$$

Evaluating at $(-i, i)$ and taking the real part gives

$$\begin{aligned} &\frac{4371}{128}\zeta(5) - \frac{349}{3072}\pi^4 \ln 2 + \frac{7}{128}\pi^2 \ln^3 2 - \frac{3}{64}\ln^5 2 \\ &\quad = 36\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 36\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] \\ &\quad + 4\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 18\mathrm{Li}_5\left[\frac{1}{2}\right] - \frac{9}{16}\mathrm{Li}_5\left[-\frac{1}{4}\right]. \end{aligned} \tag{2}$$

Evaluating the identity at $(-1, i)$ and taking the real part gives

$$\begin{aligned} &\frac{279}{8}\zeta(5) - \frac{977}{6144}\pi^4 \ln 2 + \frac{97}{768}\pi^2 \ln^3 2 - \frac{15}{128}\ln^5 2 \\ &\quad = 2\mathrm{ReLi}_5\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 36\mathrm{ReLi}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\ &\quad - 32\mathrm{ReLi}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] + 37\mathrm{Li}_5\left[\frac{1}{2}\right] + \frac{7}{8}\mathrm{Li}_5\left[-\frac{1}{4}\right]. \end{aligned} \tag{3}$$

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Broadhurst proved (Eq. 68 of [1] written out) that

$$\begin{aligned}\frac{31}{32}\zeta(5)-\frac{343}{99360}\pi^4\ln 2+\frac{5}{2484}\pi^2\ln^3 2-\frac{2}{1035}\ln^5 2 \\ =\frac{128}{69}\mathrm{ReLi}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]-\frac{20}{69}\mathrm{Li}_5\left[\frac{1}{2}\right].\end{aligned} \tag{4}$$

Solving Eqs. (1), (2), (3) and (4) simultaneously for $\zeta(5)$, $\pi^4\ln 2$, $\pi^2\ln^3 2$ and $\ln^5 2$ and using the identity

$$\mathrm{Re}\ \mathrm{Li}_5\left[pe^{iq}\right]=\sum_{k=1}^{\infty}\frac{p^k\cos(kq)}{k^5}\quad p\in[0,1],q\in[0,2\pi],$$

to write each polylogarithm term in the expressions as a BBP-type formula of length 120 and base 2^{60} and forming the required linear combination in each case gives *directly* the degree 5 binary BBP-type formulas listed in the BBP Compendium [2] for $\zeta(5)$, $\pi^4\ln 2$, $\pi^2\ln^3 2$ and $\ln^5 2$.

References

[1] David J. Broadhurst, "Polylogarithmic Ladders, Hypergeometric Series and the Ten Millionth Digits of $\zeta(3)$ and $\zeta(5)$," March 1998, available from `http://www.arxiv.org/abs/math/9803067`.

[2] D. H. Bailey, "A compendium of bbp-type formulas for mathematical constants,", October 2010, available from `http://crd.lbl.gov/~dhbailey/dhbpapers/bbp-formulas.pdf`.