Binary BBP type formulas for the arctangents of powers of the golden ratio

Kunle Adegoke, April 29, 2013

Proposition

Denote $\phi = (\sqrt{5}+1)/2$ (golden ratio). Then the following BBP type formulas (in Bailey's P notation) hold:

$$\tan^{-1}\phi = \frac{1}{16}P(1,16,8,(8,16,4,0,-2,-4,-1,0)) \tag{1}$$

$$\tan^{-1}\phi^3 = \frac{1}{8}P(1,16,8,(8,4,4,0,-2,-1,-1,0)) \tag{2}$$

$$\tan^{-1}\phi^5 = \frac{1}{16}P(1,16,8,(8,32,4,0,-2,-8,-1,0)) \tag{3}$$

$$\begin{aligned}\tan^{-1}\phi^7 = \frac{3}{4096}P(1,2^{12},24,(&2048,0,1024,0,-512,-1024,-256,\\ &0,128,0,64,0,-32,0,-16,0,8,16,4,0,-2,0,-1,0))\end{aligned} \tag{4}$$

Proof

Evaluating the trigonometric identity

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\,, \quad 0 \le x, y < 1\,,$$

at coordinates $(1/\phi, 1/\phi)$, $(1/\phi, 1/\phi^3)$, $(1/\phi^3, 1/\phi^5)$ and $(1/\phi^5, 1/\phi^7)$, and taking note of the identity $\phi^n = \phi F_n + F_{n-1}$, where F_k is the k^{th} Fibonacci number, it is straightforward to establish the following identities

$$\tan^{-1}\frac{1}{\phi} = \frac{1}{2}\tan^{-1}2 = \tan^{-1}1 - \frac{1}{2}\tan^{-1}\frac{1}{2}\,, \tag{5}$$

$$\tan^{-1}\frac{1}{\phi} + \tan^{-1}\frac{1}{\phi^3} = \tan^{-1}1\,, \tag{6}$$

$$\tan^{-1}\frac{1}{\phi^3} + \tan^{-1}\frac{1}{\phi^5} = \tan^{-1}\frac{1}{3} \tag{7}$$

and

$$\tan^{-1}\frac{1}{\phi^5}+\tan^{-1}\frac{1}{\phi^7}=\tan^{-1}\frac{1}{8}\,, \tag{8}$$

from which follow that

$$\tan^{-1}\phi=\tan^{-1}1+\frac{1}{2}\tan^{-1}\frac{1}{2}\,, \tag{9}$$

$$\tan^{-1}\phi^3=2\tan^{-1}1-\frac{1}{2}\tan^{-1}\frac{1}{2}\,, \tag{10}$$

$$\tan^{-1}\phi^5=\tan^{-1}1+\frac{3}{2}\tan^{-1}\frac{1}{2} \tag{11}$$

and

$$\tan^{-1}\phi^7=3\tan^{-1}1-\frac{3}{2}\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{8}\,. \tag{12}$$

The following formulas were derived in reference [1]:

$$\tan^{-1}\frac{1}{u}=\frac{1}{u^3}P(1,u^4,4,(u^2,0,-1,0))\,, \tag{13}$$

$$\tan^{-1}\left(\frac{1}{2u-1}\right)=\frac{1}{16u^7}P(1,16u^8,8,(8u^6,8u^5,4u^4,0,-2u^2,-2u,-1,0)) \tag{14}$$

and

$$\tan^{-1}\left(\frac{1}{2u+1}\right)=\frac{1}{16u^7}P(1,16u^8,8,(8u^6,-8u^5,4u^4,0,-2u^2,2u,-1,0))\,. \tag{15}$$

Using $u=2$ in Eq. (13) and $u=1$ in Eq. (14), and forming the indicated linear combinations, Eqs. (9)–(11) give rise to the following BBP type formulas:

$$\tan^{-1}\phi=\frac{1}{16}P(1,16,8,(8,16,4,0,-2,-4,-1,0))\,, \tag{16}$$

$$\tan^{-1}(\phi^3)=\frac{1}{8}P(1,16,8,(8,4,4,0,-2,-1,-1,0)) \tag{17}$$

and

$$\tan^{-1}(\phi^5) = \frac{1}{16} P(1, 16, 8, (8, 32, 4, 0, -2, -8, -1, 0)) . \tag{18}$$

Using $u = 1$ in Eq. (14) and $u = 2$ in Eq. (13) and expanding both series to base 2^{12}, length 24, and using $u = 8$ in Eq. (13) and finally forming the indicated linear combination gives the BBP type formula for $\tan^{-1}\phi^7$ as

$$\begin{aligned} \tan^{-1}(\phi^7) = \frac{3}{4096} P(1, 2^{12}, 24, (&2048, 0, 1024, 0, -512, -1024, -256, \\ &0, 128, 0, 64, 0, -32, 0, -16, 0, 8, 16, 4, 0, -2, 0, -1, 0)) . \end{aligned} \tag{19}$$

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References

[1] Kunle Adegoke A non-PSLQ route to BBP-type formulas. *Journal of Mathematics Research*, 2(2):56–64, 2010.