

# A novel approach to the discovery of ternary BBP-type formulas for polylogarithm constants

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**Abstract:** Using clear and straightforward approaches, we prove new ternary (base 3) digit extraction BBP-type formulas for polylogarithm constants. Some known results are also rediscovered in a more direct and elegant manner. An hitherto unproved degree 4 ternary formula is also proved. Finally, a couple of ternary zero relations are established, which prove two known but hitherto unproved formulas.

**Keywords:** Ternary BBP-type formulas, Zero Relations, Digit extraction formulas

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## 1 Introduction

BBP-type formulas are formulas of the form

$$\alpha = \sum_{k=0}^{\infty} 1/b^k \sum_{j=1}^n a_j / (kn + j)^s$$

where  $s$ ,  $b$ ,  $n$  and  $a_j$  are integers, and  $\alpha$  is some constant. Formulas of this type were first introduced in a 1996 paper [1], where a formula of this type for  $\pi$  was given. Such formulas have the remarkable property that they permit one to calculate base- $b$  digits of the constant  $\alpha$  beginning at an arbitrary starting position, by means of a simple algorithm that requires almost no memory and (depending on how many digits are required) without the need for multiple-precision arithmetic software [2]. Such formulas also have intriguing connections to the age-old problem of understanding why the digits of various transcendental constants appear “normal” – each string of  $m$ -long digits appears, in the limit, with frequency  $1/b^m$  [3, 4, 5, 2].

While many binary BBP-type formulas are now known, only relatively few ternary (base-3) BBP-type formulas have been discovered. This present paper is concerned with the symbolic (that is, non-computer-search-based) discovery of ternary (base-3) BBP-type formulas for polylogarithm constants. The methods used here aim to complement the experimental approaches that have dominated the area. Through fundamental methods, a wide range of interesting formulas will be obtained. In most cases, the procedure for obtaining the ternary formulas shall consist mainly of evaluating a polylogarithm functional equation at indicated coordinates and noting the following identities for the

real and imaginary parts of the polylogarithm function:

$$\begin{aligned}\operatorname{Re} \operatorname{Li}_s [pe^{ix}] &= \sum_{k=1}^{\infty} \frac{p^k \cos kx}{k^s} \\ \operatorname{Im} \operatorname{Li}_s [pe^{ix}] &= \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k^s},\end{aligned}\tag{1}$$

for  $p \in [0, 1]$ ,  $x \in \mathbb{R}$  and  $s \in \mathbb{Z}^+$ . In the above equations,  $\operatorname{Li}$  is the notation for the polylogarithm function defined by

$$\operatorname{Li}_s[z] = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad |z| \leq 1.$$

When  $p = 1$  we have

$$\begin{aligned}\operatorname{Li}_{2n}[e^{ix}] &= \operatorname{Gl}_{2n}(x) + i\operatorname{Cl}_{2n}(x) \\ \operatorname{Li}_{2n+1}[e^{ix}] &= \operatorname{Cl}_{2n+1}(x) + i\operatorname{Gl}_{2n+1}(x),\end{aligned}\tag{2}$$

where  $\operatorname{Gl}$  and  $\operatorname{Cl}$  are Clausen sums [6] defined, for  $n \in \mathbb{Z}^+$  by

$$\begin{aligned}\operatorname{Cl}_{2n}(x) &= \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2n}}, & \operatorname{Cl}_{2n+1}(x) &= \sum_{k=1}^{\infty} \frac{\cos kx}{k^{2n+1}} \\ \operatorname{Gl}_{2n}(x) &= \sum_{k=1}^{\infty} \frac{\cos kx}{k^{2n}}, & \operatorname{Gl}_{2n+1}(x) &= \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2n+1}}.\end{aligned}\tag{3}$$

We shall find the following formulas useful:

$$\begin{aligned}\operatorname{Gl}_{2n}(x) &= (-1)^{1+[n/2]} 2^{n-1} \pi^n \operatorname{B}_n(x/2\pi)/n! \\ \frac{1}{m^{n-1}} \operatorname{Cl}_n(mx) &= \sum_{r=0}^{m-1} \operatorname{Cl}_n(x + 2\pi r/m).\end{aligned}\tag{4}$$

Here  $[n/2]$  denotes the integer part of  $n/2$  and  $\operatorname{B}_n$  are the Bernoulli polynomials defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{\operatorname{B}_n(x)t^n}{n!}.$$

## 2 Degree 1 Ternary BBP-type Formulas

In reference [7], several degree 1 BBP-type formulas in general bases are proven. In many of the formulas, ternary formulas may be readily obtained by writing the base in each case as a power of 3.

Here we now present a couple of interesting degree 1 ternary formulas.

The following identities are easily verified:

$$\operatorname{Li}_1 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right] = \frac{1}{2} \ln 3 + \frac{i\pi}{6}$$

and

(5)

$$\operatorname{Li}_1 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{2} \right) \right] = \frac{1}{2} \ln 3 - \ln 2 + \frac{i\pi}{6}.$$

We therefore have the formulas:

$$\ln 2 = \operatorname{Re} \operatorname{Li}_1 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right] - \operatorname{Re} \operatorname{Li}_1 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{2} \right) \right],$$
(6)

$$\ln 3 = 2 \operatorname{Re} \operatorname{Li}_1 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right]$$
(7)

and

$$\pi = 6 \operatorname{Im} \operatorname{Li}_1 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right] = 6 \operatorname{Im} \operatorname{Li}_1 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{2} \right) \right].$$
(8)

It is also straightforward to verify that:

$$\ln 2 = \operatorname{Li}_1 \left[ \frac{1}{3} \right] - \operatorname{Li}_1 \left[ -\frac{1}{3} \right]$$

and

(9)

$$\ln 3 = 2 \operatorname{Li}_1 \left[ \frac{1}{3} \right] - \operatorname{Li}_1 \left[ -\frac{1}{3} \right].$$

Based on the above formulas, we are now ready to derive explicit BBP-type formulas for  $\ln 2$ ,  $\ln 3$  and  $\pi$ .

## 2.1 Ternary formulas for $\ln 2$

Using the first of Eq. (1), we note that

$$\begin{aligned} \operatorname{Re} \operatorname{Li}_1 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right] &= \sum_{k=1}^{\infty} \left( \frac{1}{\sqrt{3}} \right)^k \frac{\cos(k\pi/6)}{k} \\ &= \frac{1}{2 \cdot 3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^6}{12k+1} + \frac{3^5}{12k+2} \right. \\ &\quad - \frac{3^4}{12k+4} - \frac{3^4}{12k+5} - \frac{2 \cdot 3^3}{12k+6} - \frac{3^3}{12k+7} \\ &\quad \left. - \frac{3^2}{12k+8} + \frac{3}{12k+10} + \frac{3}{12k+11} + \frac{2}{12k+12} \right] \end{aligned}$$
(10)

and

$$\operatorname{Re} \operatorname{Li}_1 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{2} \right) \right] = \frac{1}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ -\frac{3^5}{12k+2} + \frac{3^4}{12k+4} - \frac{3^3}{12k+6} + \frac{3^2}{12k+8} - \frac{3}{12k+10} + \frac{2}{12k+12} \right]. \quad (11)$$

Subtracting Eq. (11) from Eq. (10) in accordance with Eq. (6), we obtain the following ternary BBP-type formula for  $\ln 2$ :

$$\begin{aligned} \ln 2 = & \frac{1}{2 \cdot 3^5} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^5}{12k+1} + \frac{3^5}{12k+2} \right. \\ & - \frac{3^4}{12k+4} - \frac{3^3}{12k+5} - \frac{3^2}{12k+7} - \frac{3^2}{12k+8} \\ & \left. + \frac{3}{12k+10} + \frac{1}{12k+11} \right]. \end{aligned} \quad (12)$$

Note that an alternating version of Eq. (12), using the same scheme, is

$$\ln 2 = \frac{1}{18} \sum_{k=0}^{\infty} \left( -\frac{1}{27} \right)^k \left[ \frac{9}{6k+1} + \frac{9}{6k+2} - \frac{3}{6k+4} - \frac{1}{6k+5} \right].$$

From the first of Eq. (9), we can obtain yet another ternary formula for  $\ln 2$ , as follows:

$$\operatorname{Li}_1 \left[ \frac{1}{3} \right] = \frac{1}{9} \sum_{k=0}^{\infty} \frac{1}{9^k} \left[ \frac{3}{2k+1} + \frac{1}{2k+2} \right] \quad (13)$$

and

$$\operatorname{Li}_1 \left[ -\frac{1}{3} \right] = \frac{1}{9} \sum_{k=0}^{\infty} \frac{1}{9^k} \left[ -\frac{3}{2k+1} + \frac{1}{2k+2} \right] \quad (14)$$

Subtracting Eq. (14) from Eq. (13) in accordance with the first of Eq. (9), we obtain

$$\ln 2 = \frac{3}{2} \sum_{k=0}^{\infty} \frac{1}{9^k} \left[ \frac{1}{2k+1} \right], \quad (15)$$

which is listed in the BBP Compendium as formula (48).

## 2.2 Ternary formulas for $\ln 3$

Using Eq. (10) and Eq. (7), we obtain the following ternary BBP-type formula for  $\ln 3$ :

$$\begin{aligned} \ln 3 = & \frac{1}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^6}{12k+1} + \frac{3^5}{12k+2} \right. \\ & - \frac{3^4}{12k+4} - \frac{3^4}{12k+5} - \frac{2 \cdot 3^3}{12k+6} - \frac{3^3}{12k+7} \\ & \left. - \frac{3^2}{12k+8} + \frac{3}{12k+10} + \frac{3}{12k+11} + \frac{2}{12k+12} \right]. \end{aligned} \quad (16)$$

An alternating version of the above formula is

$$\ln 3 = \frac{1}{27} \sum_{k=0}^{\infty} \left(-\frac{1}{27}\right)^k \left[ \frac{27}{6k+1} + \frac{9}{6k+2} - \frac{3}{6k+4} - \frac{3}{6k+5} - \frac{2}{6k+6} \right].$$

Combining Eq. (13) and Eq. (14) according to the second equality of Eq. (9) we obtain another ternary BBP-type formula for  $\ln 3$  as :

$$\ln 3 = \frac{1}{9} \sum_{k=0}^{\infty} \frac{1}{9^k} \left[ \frac{9}{2k+1} + \frac{1}{2k+2} \right], \quad (17)$$

which is Formula (51) of the Compendium.

### 2.3 Ternary formulas for $\pi\sqrt{3}$

From Eq. (8), we immediately obtain the ternary BBP-type formulas

$$\begin{aligned} \pi\sqrt{3} = \frac{1}{3^4} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} & \left[ \frac{3^5}{12k+1} + \frac{3^5}{12k+2} + \frac{2 \cdot 3^4}{12k+3} \right. \\ & + \frac{3^4}{12k+4} + \frac{3^3}{12k+5} - \frac{3^2}{12k+7} - \frac{3^2}{12k+8} \\ & \left. - \frac{2 \cdot 3}{12k+9} - \frac{3}{12k+10} - \frac{1}{12k+11} \right] \end{aligned} \quad (18)$$

and

$$\begin{aligned} \pi\sqrt{3} = \frac{2}{3^4} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} & \left[ \frac{3^5}{12k+1} - \frac{3^4}{12k+3} + \frac{3^3}{12k+5} \right. \\ & \left. - \frac{3^2}{12k+7} + \frac{3}{12k+9} - \frac{1}{12k+11} \right]. \end{aligned} \quad (19)$$

An alternating version of Eq. (19) is

$$\pi\sqrt{3} = 6 \sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k \left[ \frac{1}{2k+1} \right]. \quad (20)$$

### 2.4 Ternary Zero Relations

Eq. (15) may be rewritten in base  $3^6$ , length 12 as

$$\ln 2 = \frac{4}{3^5} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^4}{12k+2} + \frac{3^2}{12k+6} + \frac{1}{12k+10} \right] \quad (21)$$

Subtracting Eq. (21) from Eq. (12), we obtain the following ternary zero relation:

$$\begin{aligned}
0 = \sum_{k=0}^{\infty} \frac{1}{3^{6k}} & \left[ \frac{3^5}{12k+1} - \frac{5 \cdot 3^4}{12k+2} - \frac{3^4}{12k+4} \right. \\
& - \frac{3^3}{12k+5} - \frac{2^3 \cdot 3^2}{12k+6} - \frac{3^2}{12k+7} - \frac{3^2}{12k+8} \\
& \left. - \frac{5}{12k+10} + \frac{1}{12k+11} \right]. \tag{22}
\end{aligned}$$

Note that

$$\text{Eq. (22)} = \text{Compendium formula (104)} - \text{Compendium formula (103)} = 0. \tag{23}$$

Subtracting Eq. (18) from Eq. (19), we obtain the zero relation:

$$\begin{aligned}
0 = \sum_{k=0}^{\infty} \frac{1}{3^{6k}} & \left[ \frac{243}{12k+1} - \frac{243}{12k+2} - \frac{324}{12k+3} \right. \\
& - \frac{81}{12k+4} + \frac{27}{12k+5} - \frac{9}{12k+7} + \frac{9}{12k+8} \\
& \left. + \frac{12}{12k+9} + \frac{3}{12k+10} - \frac{1}{12k+11} \right]. \tag{24}
\end{aligned}$$

Note also that

$$\text{Eq. (24)} = \text{Compendium formula (104)} + \text{Compendium formula (103)} = 0 \tag{25}$$

Eqs. (23) and (25) therefore establish that

$$\begin{aligned}
& \text{Compendium formula(103)} = 0 \\
& \text{and} \\
& \text{Compendium formula(104)} = 0.
\end{aligned} \tag{26}$$

Thus the hitherto unproved formulas (103) and (104) in the BBP Compendium are now proved.

### 3 Degree 2 Ternary BBP-type Formulas

The dilogarithm reflection formula (Eq. A.2.1.7 of [6]) is

$$\frac{\pi^2}{6} - \ln x \ln(1-x) = \text{Li}_2[x] + \text{Li}_2[1-x].$$

Putting  $x = -\exp(i\pi/3)$  in the above equation and taking real and imaginary parts gives

$$\frac{5\pi^2}{72} - \frac{1}{8} \ln^2 3 = \text{Re Li}_2 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right] \tag{27}$$

and

$$\frac{2}{3}\text{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\pi \ln 3}{12} = \text{Im Li}_2\left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{6}\right)\right]. \quad (28)$$

A two-variable functional equation for dilogarithms, due to Kummer (Eq. A.2.1.19 of [6]) is

$$\begin{aligned} \text{Li}_2\left[\frac{x(1-y)^2}{y(1-x)^2}\right] &= \text{Li}_2\left[-\frac{x(1-y)}{(1-x)}\right] + \text{Li}_2\left[-\frac{(1-y)}{y(1-x)}\right] \\ &+ \text{Li}_2\left[\frac{x(1-y)}{y(1-x)}\right] + \text{Li}_2\left[\frac{1-y}{1-x}\right] + \frac{1}{2}\ln^2 y. \end{aligned} \quad (29)$$

Choosing  $x = -\exp(i\pi/3)$  and  $y = \exp(i\pi/3)$  in Eq. (29) gives

$$\frac{\pi^2}{12} - \frac{\ln^2 3}{4} = \text{Li}_2\left[\frac{1}{3}\right] - \frac{1}{2}\text{Li}_2\left[-\frac{1}{3}\right]. \quad (30)$$

Note that the choice of  $x = -1$  and  $y = 1/3$  gives the same result.

Another two-variable functional equation for dilogarithms, due to Abel (Eq. A.2.1.16 of [6]) is

$$\begin{aligned} \text{Li}_2\left[\frac{x}{1-x} \cdot \frac{y}{1-y}\right] &= \text{Li}_2\left[\frac{x}{(1-y)}\right] + \text{Li}_2\left[\frac{y}{(1-x)}\right] \\ &- \text{Li}_2[x] - \text{Li}_2[y] - \ln(1-x)\ln(1-y). \end{aligned} \quad (31)$$

Choosing  $x = -\exp(i\pi/3)$  and  $y = \exp(-i\pi/3)$  in Eq. (31) and taking imaginary parts gives

$$\frac{5}{2}\text{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\pi \ln 3}{4} = 3\text{Im Li}_2\left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{2}\right)\right]. \quad (32)$$

### 3.1 Ternary Formula for $\pi^2$

Solving Eq. (27) and Eq. (30) for  $\pi^2$ , we obtain

$$\pi^2 = 36\text{Re Li}_2\left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{6}\right)\right] - 18\text{Li}_2\left[\frac{1}{3}\right] + 9\text{Li}_2\left[-\frac{1}{3}\right]. \quad (33)$$

Writing

$$\begin{aligned} \text{Re Li}_2\left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{6}\right)\right] &= \frac{1}{2 \cdot 3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^6}{(12k+1)^2} + \frac{3^5}{(12k+2)^2} \right. \\ &- \frac{3^4}{(12k+4)^2} - \frac{3^4}{(12k+5)^2} - \frac{3^3}{(12k+6)^2} \\ &- \frac{3^3}{(12k+7)^2} - \frac{3^2}{(12k+8)^2} \\ &\left. + \frac{3}{(12k+10)^2} + \frac{2}{(12k+12)^2} \right] \end{aligned} \quad (34)$$

and

$$\begin{aligned} \operatorname{Li}_2 \left[ \pm \frac{1}{3} \right] &= \frac{2^2}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \pm \frac{3^5}{(12k+2)^2} + \frac{3^4}{(12k+4)^2} \right. \\ &\quad \pm \frac{3^3}{(12k+6)^2} + \frac{3^2}{(12k+8)^2} \\ &\quad \left. \pm \frac{3}{(12k+10)^2} + \frac{1}{(12k+12)^2} \right], \end{aligned} \quad (35)$$

and combining them according to Eq. (33) gives the ternary BBP-type formula for  $\pi^2$  as

$$\begin{aligned} \pi^2 &= \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^5}{(12k+1)^2} - \frac{5 \cdot 3^4}{(12k+2)^2} \right. \\ &\quad - \frac{3^4}{(12k+4)^2} - \frac{3^3}{(12k+5)^2} - \frac{2^3 \cdot 3^2}{(12k+6)^2} \\ &\quad - \frac{3^2}{(12k+7)^2} - \frac{3^2}{(12k+8)^2} \\ &\quad \left. - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2} \right]. \end{aligned} \quad (36)$$

Incidentally, Eq. (36) is Formula (57) of the Compendium.

### 3.2 Ternary Formula for $\ln^2 3$

Solving Eq. (27) and Eq. (30) for  $\ln^2 3$ , we obtain

$$\ln^2 3 = 12 \operatorname{Re} \operatorname{Li}_2 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right] - 10 \operatorname{Li}_2 \left[ \frac{1}{3} \right] + 5 \operatorname{Li}_2 \left[ -\frac{1}{3} \right]. \quad (37)$$

Using Eqs. (34) and (35) above in Eq. (37), we obtain the ternary BBP-type formula for  $\ln^2 3$  as

$$\begin{aligned} \ln^2 3 &= \frac{1}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{2 \cdot 3^7}{(12k+1)^2} - \frac{2 \cdot 3^8}{(12k+2)^2} \right. \\ &\quad - \frac{2 \cdot 13 \cdot 3^4}{(12k+4)^2} - \frac{2 \cdot 3^5}{(12k+5)^2} - \frac{2^3 \cdot 3^5}{(12k+6)^2} \\ &\quad - \frac{2 \cdot 3^4}{(12k+7)^2} - \frac{2 \cdot 13 \cdot 3^2}{(12k+8)^2} \\ &\quad \left. - \frac{2 \cdot 3^4}{(12k+10)^2} + \frac{2 \cdot 3^2}{(12k+11)^2} - \frac{8}{(12k+12)^2} \right], \end{aligned} \quad (38)$$

which is Formula (58) of the Compendium.



### 3.3 Ternary Formula for $\pi\sqrt{3}\ln 3$

Solving Eqs. (28) and (32) for  $\pi\ln 3$  gives

$$\pi\ln 3 = 48 \operatorname{Im} \operatorname{Li}_2 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{2} \right) \right] - 60 \operatorname{Im} \operatorname{Li}_2 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right]. \quad (39)$$

Now

$$\begin{aligned} \operatorname{Im} \operatorname{Li}_2 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right] &= \frac{\sqrt{3}}{54} \sum_{k=0}^{\infty} \left( -\frac{1}{27} \right)^k \left[ \frac{9}{(6k+1)^2} + \frac{9}{(6k+2)^2} \right. \\ &\quad \left. + \frac{6}{(12k+3)^2} + \frac{3}{(12k+4)^2} + \frac{1}{(6k+5)^2} \right] \end{aligned} \quad (40)$$

and

$$\operatorname{Im} \operatorname{Li}_2 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{2} \right) \right] = \frac{\sqrt{3}}{27} \sum_{k=0}^{\infty} \left( -\frac{1}{27} \right)^k \left[ \frac{9}{(6k+1)^2} - \frac{3}{(6k+3)^2} + \frac{1}{(6k+5)^2} \right] \quad (41)$$

Using Eqs. (40) and (41) in Eq. (39) leads to the ternary BBP-type formula

$$\begin{aligned} \pi\sqrt{3}\ln 3 &= 2 \sum_{k=0}^{\infty} \left( -\frac{1}{27} \right)^k \left[ \frac{9}{(6k+1)^2} - \frac{15}{(6k+2)^2} - \frac{18}{(6k+3)^2} \right. \\ &\quad \left. - \frac{5}{(6k+4)^2} + \frac{1}{(6k+5)^2} \right]. \end{aligned} \quad (42)$$

### 3.4 Ternary Formula for $\sqrt{3}\operatorname{Cl}_2(\pi/3)$

Solving Eqs. (28) and (32) for  $\operatorname{Cl}_2(\pi/3)$  gives

$$\operatorname{Cl}_2 \left( \frac{\pi}{3} \right) = 6 \operatorname{Im} \operatorname{Li}_2 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{2} \right) \right] - 6 \operatorname{Im} \operatorname{Li}_2 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right]. \quad (43)$$

Using Eqs. (40) and (41) in Eq. (43) leads to the ternary BBP-type formula

$$\begin{aligned} \sqrt{3}\operatorname{Cl}_2 \left( \frac{\pi}{3} \right) &= \frac{1}{3} \sum_{k=0}^{\infty} \left( -\frac{1}{27} \right)^k \left[ \frac{9}{(6k+1)^2} - \frac{9}{(6k+2)^2} - \frac{12}{(6k+3)^2} \right. \\ &\quad \left. - \frac{3}{(6k+4)^2} + \frac{1}{(6k+5)^2} \right]. \end{aligned} \quad (44)$$

## 4 Degree 3 Ternary BBP-type Formulas

A functional identity for trilogarithms (Eq. A.2.6.10 of [6]) is

$$\begin{aligned} \operatorname{Li}_3 \left[ \frac{1-x}{1+x} \right] - \operatorname{Li}_3 \left[ \frac{x-1}{x+1} \right] &= 2 \operatorname{Li}_3 [1-x] + 2 \operatorname{Li}_3 \left[ \frac{1}{1+x} \right] \\ &\quad - \frac{1}{2} \operatorname{Li}_3 [1-x^2] - \frac{7}{4} \zeta(3) \\ &\quad + \frac{\pi^2}{6} \ln(1+x) - \frac{1}{3} \ln^3(1+x). \end{aligned} \quad (45)$$

The use of  $x = 2$  in the above equation gives

$$\frac{13}{6} \zeta(3) - \frac{1}{6} \pi^2 \ln 3 + \frac{1}{6} \ln^3 3 = 2 \operatorname{Li}_3 \left[ \frac{1}{3} \right] - \operatorname{Li}_3 \left[ -\frac{1}{3} \right]. \quad (46)$$

Putting  $x = \exp i\pi/3$  in the functional equation and taking real and imaginary parts gives

$$\frac{13}{18} \zeta(3) - \frac{5}{144} \pi^2 \ln 3 + \frac{1}{48} \ln^3 3 = \operatorname{Re} \operatorname{Li}_3 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right] \quad (47)$$

and

$$\begin{aligned} \frac{29}{1296} \pi^3 - \frac{1}{48} \pi \ln^2 3 \\ = 4 \operatorname{Im} \operatorname{Li}_3 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{2} \right) \right] - 5 \operatorname{Im} \operatorname{Li}_3 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right]. \end{aligned} \quad (48)$$

Using

$$\begin{aligned} \operatorname{Li}_3 \left[ \pm \frac{1}{3} \right] &= \frac{1}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \pm \frac{3^5}{(6k+1)^3} + \frac{3^4}{(6k+2)^3} \pm \frac{3^3}{(6k+3)^3} \right. \\ &\quad \left. + \frac{3^2}{(6k+4)^3} \pm \frac{3}{(6k+5)^3} + \frac{1}{(6k+6)^3} \right] \end{aligned} \quad (49)$$

leads to the ternary BBP-type formula

$$\begin{aligned} 13\zeta(3) - \pi^2 \ln 3 + \ln^3 3 \\ = \frac{2}{3^5} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^6}{(6k+1)^3} + \frac{3^4}{(6k+2)^3} + \frac{3^4}{(6k+3)^3} \right. \\ \left. + \frac{3^2}{(6k+4)^3} + \frac{3^2}{(6k+5)^3} + \frac{1}{(6k+6)^3} \right]. \end{aligned} \quad (50)$$

A shorter version (length 2) of the above formula is

$$13\zeta(3) - \pi^2 \ln 3 + \ln^3 3 = \frac{2}{3} \sum_{k=0}^{\infty} \frac{1}{9^k} \left[ \frac{9}{(2k+1)^3} + \frac{1}{(2k+2)^3} \right]. \quad (51)$$

The ternary BBP-type formula that results from Eq. (47) is discussed elsewhere [8].

Next we obtain the ternary BBP-type formula that results from Eq. (48).

$$\begin{aligned} \operatorname{Im} \operatorname{Li}_3 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{2} \right) \right] &= \frac{\sqrt{3}}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^5}{(12k+1)^3} - \frac{3^4}{(12k+3)^3} \right. \\ &\quad \left. + \frac{3^3}{(12k+5)^3} - \frac{3^2}{(12k+7)^3} + \frac{3}{(12k+9)^3} - \frac{1}{(12k+11)^3} \right] \end{aligned} \quad (52)$$

and

$$\begin{aligned} \operatorname{Im} \operatorname{Li}_3 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right] &= \frac{\sqrt{3}}{2 \cdot 3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^5}{(12k+1)^3} + \frac{3^5}{(12k+2)^3} + \frac{2 \cdot 3^4}{(12k+3)^3} \right. \\ &\quad + \frac{3^4}{(12k+4)^3} + \frac{3^3}{(12k+5)^3} - \frac{3^2}{(12k+7)^3} \\ &\quad \left. - \frac{3^2}{(12k+8)^3} - \frac{2 \cdot 3}{(12k+9)^3} - \frac{3}{(12k+10)^3} - \frac{1}{(12k+11)^3} \right] \end{aligned} \quad (53)$$

Combining Eqs. (52) and (53) according to the prescription of Eq. (48), we arrive at

$$\begin{aligned} \frac{29}{1296} \pi^3 \sqrt{3} - \frac{1}{48} \pi \sqrt{3} \ln^2 3 &= \frac{1}{2 \cdot 3^5} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^6}{(12k+1)^3} - \frac{5 \cdot 3^5}{(12k+2)^3} \right. \\ &\quad - \frac{2 \cdot 3^6}{(12k+3)^3} - \frac{5 \cdot 3^4}{(12k+4)^3} + \frac{3^4}{(12k+5)^3} - \frac{3^3}{(12k+7)^3} \\ &\quad \left. + \frac{5 \cdot 3^2}{(12k+8)^3} + \frac{2 \cdot 3^3}{(12k+9)^3} + \frac{5 \cdot 3}{(12k+10)^3} - \frac{3}{(12k+11)^3} \right]. \end{aligned} \quad (54)$$

## 5 Degree 4 Ternary BBP-type Formulas

A two-variable functional equation for degree 4 polylogarithms (Eq. A.2.7.40 of [6]) reads

$$\begin{aligned} &\operatorname{Li}_4 \left[ -\frac{x^2 y \eta}{\xi} \right] + \operatorname{Li}_4 \left[ -\frac{y^2 x \xi}{\eta} \right] + \operatorname{Li}_4 \left[ \frac{x^2 y}{\eta^2 \xi} \right] + \operatorname{Li}_4 \left[ \frac{y^2 x}{\xi^2 \eta} \right] \\ &= 6 \operatorname{Li}_4 [xy] + 6 \operatorname{Li}_4 \left[ \frac{xy}{\eta \xi} \right] + 6 \operatorname{Li}_4 \left[ -\frac{xy}{\eta} \right] + 6 \operatorname{Li}_4 \left[ -\frac{xy}{\xi} \right] \\ &+ 3 \operatorname{Li}_4 [x\eta] + 3 \operatorname{Li}_4 [y\xi] + 3 \operatorname{Li}_4 \left[ \frac{x}{\eta} \right] + 3 \operatorname{Li}_4 \left[ \frac{y}{\xi} \right] + 3 \operatorname{Li}_4 \left[ -\frac{x\eta}{\xi} \right] \\ &+ 3 \operatorname{Li}_4 \left[ -\frac{y\xi}{\eta} \right] + 3 \operatorname{Li}_4 \left[ -\frac{x}{\eta \xi} \right] + 3 \operatorname{Li}_4 \left[ -\frac{y}{\eta \xi} \right] - 6 \operatorname{Li}_4 [x] \\ &- 6 \operatorname{Li}_4 [y] - 6 \operatorname{Li}_4 \left[ -\frac{x}{\xi} \right] - 6 \operatorname{Li}_4 \left[ -\frac{y}{\eta} \right] + 3/2 \ln^2 \xi \ln^2 \eta, \end{aligned} \quad (55)$$

where  $\xi = 1 - x$ ,  $\eta = 1 - y$ .

Putting  $x = -\exp(i\pi/3)$  and  $y = \exp(i\pi/3)$  in Eq. (55), simplifying and taking real and imaginary parts, we obtain

$$\begin{aligned}
& -12 \operatorname{Re} \operatorname{Li}_4 \left[ \frac{1}{\sqrt{3}} e^{i\pi/2} \right] - 3 \operatorname{Re} \operatorname{Li}_4 \left[ \frac{1}{\sqrt{3}} e^{i\pi/6} \right] \\
& + \operatorname{Li}_4 \left[ \frac{1}{3} \right] + \frac{1}{4} \operatorname{Li}_4 \left[ -\frac{1}{3} \right] \\
& = -\frac{127\pi^4}{10368} + \frac{1}{64} \pi^2 \ln^2 3 - \frac{5}{384} \ln^4 3
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
& -12 \operatorname{Im} \operatorname{Li}_4 \left[ \frac{1}{\sqrt{3}} e^{i\pi/2} \right] + 15 \operatorname{Im} \operatorname{Li}_4 \left[ \frac{1}{\sqrt{3}} e^{i\pi/6} \right] \\
& = \frac{29}{864} \pi^3 \ln 3 - \frac{1}{96} \pi \ln^3 3 - \frac{11}{3} \operatorname{Cl}_4 \left( \frac{\pi}{3} \right).
\end{aligned} \tag{57}$$

First we proceed to obtain the BBP-type formula invoked by Eq. (56).

Now,

$$\begin{aligned}
\operatorname{Re} \operatorname{Li}_4 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{2} \right) \right] &= \frac{1}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ -\frac{3^5}{(12k+2)^4} + \frac{3^4}{(12k+4)^4} \right. \\
& \quad \left. - \frac{3^3}{(12k+6)^4} + \frac{3^2}{(12k+8)^4} - \frac{3}{(12k+10)^4} + \frac{1}{(12k+12)^4} \right]
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
\operatorname{Re} \operatorname{Li}_4 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right] &= \frac{1}{2 \cdot 3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^6}{(12k+1)^4} + \frac{3^5}{(12k+2)^4} \right. \\
& \quad - \frac{3^4}{(12k+4)^4} - \frac{3^4}{(12k+5)^4} - \frac{2 \cdot 3^3}{(12k+6)^4} \\
& \quad - \frac{3^3}{(12k+7)^4} - \frac{3^2}{(12k+8)^4} + \frac{3}{(12k+10)^4} \\
& \quad \left. + \frac{2}{(12k+12)^4} \right].
\end{aligned} \tag{59}$$

Also

$$\begin{aligned}
\operatorname{Li}_4 \left[ \pm \frac{1}{3} \right] &= \frac{2^4}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \pm \frac{3^5}{(12k+2)^4} + \frac{3^4}{(12k+4)^4} \right. \\
& \quad \pm \frac{3^3}{(12k+6)^4} + \frac{3^2}{(12k+8)^4} \pm \frac{3}{(12k+10)^4} \\
& \quad \left. + \frac{1}{(12k+12)^4} \right].
\end{aligned} \tag{60}$$

Combining Eqs. (58), (59) and (60) according to Eq. (56), we obtain the following degree 4 ternary BBP-type formula:

$$\begin{aligned} \frac{127\pi^4}{5184} - \frac{\pi^2 \ln^2 3}{32} + \frac{5 \ln^4 3}{192} &= \frac{1}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^7}{(12k+1)^4} - \frac{5 \cdot 3^7}{(12k+2)^4} \right. \\ &\quad - \frac{19 \cdot 3^4}{(12k+4)^4} - \frac{3^5}{(12k+5)^4} - \frac{2 \cdot 3^6}{(12k+6)^4} \\ &\quad - \frac{3^4}{(12k+7)^4} - \frac{19 \cdot 3^2}{(12k+8)^4} - \frac{5 \cdot 3^3}{(12k+10)^4} \\ &\quad \left. + \frac{3^2}{(12k+11)^4} - \frac{10}{(12k+12)^4} \right]. \end{aligned} \quad (61)$$

Next we obtain the BBP-type formula invoked by Eq. (57).

Writing

$$\begin{aligned} \operatorname{Im} \operatorname{Li}_4 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{2} \right) \right] \\ = \frac{\sqrt{3}}{27} \sum_{k=0}^{\infty} \left( -\frac{1}{27} \right)^k \left[ \frac{9}{(6k+1)^4} - \frac{3}{(6k+3)^4} + \frac{1}{(6k+5)^4} \right] \end{aligned} \quad (62)$$

and

$$\begin{aligned} \operatorname{Im} \operatorname{Li}_4 \left[ \frac{1}{\sqrt{3}} \exp \left( \frac{i\pi}{6} \right) \right] &= \frac{\sqrt{3}}{54} \sum_{k=0}^{\infty} \left( -\frac{1}{27} \right)^k \left[ \frac{9}{(6k+1)^4} + \frac{9}{(6k+2)^4} \right. \\ &\quad \left. + \frac{6}{(6k+3)^4} + \frac{3}{(6k+4)^4} + \frac{1}{(6k+5)^4} \right], \end{aligned} \quad (63)$$

and combining these according to Eq. (57), we obtain the following degree 4 BBP-type formula:

$$\begin{aligned} \frac{1}{\sqrt{3}} \left( 11 \operatorname{Cl}_4 \left( \frac{\pi}{3} \right) - \frac{29}{288} \pi^3 \ln 3 + \frac{\pi \ln^3 3}{32} \right) \\ = \frac{1}{2} \sum_{k=0}^{\infty} \left( -\frac{1}{27} \right)^k \left[ \frac{9}{(6k+1)^4} - \frac{15}{(6k+2)^4} \right. \\ \left. - \frac{18}{(6k+3)^4} - \frac{5}{(6k+4)^4} + \frac{1}{(6k+5)^4} \right]. \end{aligned} \quad (64)$$

It is interesting to remark that Eq. (64) was also obtained by Broadhurst [9], using the PSLQ Algorithm. We have thus found its formal proof for the first time, through Eq. (57)!

## 6 Degree 5 Ternary BBP-type Formulas

The following degree 5 polylogarithm identity is derived in [10]

$$\begin{aligned}
& \text{Li}_5 \left[ \frac{x\alpha}{y\beta} \right] + \text{Li}_5 [x\alpha y\eta] + \text{Li}_5 \left[ \frac{x\alpha\beta}{\eta} \right] + \text{Li}_5 [x\xi y\beta] + \text{Li}_5 \left[ \frac{x\xi}{y\eta} \right] \\
& + \text{Li}_5 \left[ \frac{x\xi\eta}{\beta} \right] + \text{Li}_5 \left[ \frac{\alpha y\beta}{\xi} \right] + \text{Li}_5 \left[ \frac{\alpha}{\xi y\eta} \right] + \text{Li}_5 \left[ \frac{\alpha\eta}{\xi\beta} \right] \\
& - 9 \text{Li}_5 [xy] - 9 \text{Li}_5 [x\beta] - 9 \text{Li}_5 [x\eta] - 9 \text{Li}_5 \left[ \frac{x}{y} \right] - 9 \text{Li}_5 \left[ \frac{x}{\beta} \right] \\
& - 9 \text{Li}_5 \left[ \frac{x}{\eta} \right] - 9 \text{Li}_5 [\alpha y] - 9 \text{Li}_5 [\alpha\beta] - 9 \text{Li}_5 [\alpha\eta] \\
& - 9 \text{Li}_5 \left[ \frac{\alpha}{y} \right] - 9 \text{Li}_5 \left[ \frac{\alpha}{\beta} \right] - 9 \text{Li}_5 \left[ \frac{\alpha}{\eta} \right] - 9 \text{Li}_5 [\xi y] - 9 \text{Li}_5 [\xi\beta] \\
& - 9 \text{Li}_5 [\xi\eta] - 9 \text{Li}_5 \left[ \frac{y}{\xi} \right] - 9 \text{Li}_5 \left[ \frac{\beta}{\xi} \right] - 9 \text{Li}_5 \left[ \frac{\eta}{\xi} \right] \\
& + 18 \text{Li}_5 [x] + 18 \text{Li}_5 [\alpha] + 18 \text{Li}_5 [\xi] + 18 \text{Li}_5 [y] + 18 \text{Li}_5 [\beta] \\
& + 18 \text{Li}_5 [\eta] - 18\zeta(5) = 3/10 (\ln \xi)^5 + 3/4 (\ln y - \ln x) (\ln \xi)^4 \\
& + 3/2 (3 \ln y - \ln \eta) (\ln \eta)^2 (\ln \xi)^2 + 1/2 \pi^2 (\ln \xi - 3 \ln \eta) (\ln \xi)^2 + 1/5 \pi^4 \ln \xi.
\end{aligned} \tag{65}$$

Here  $\xi = 1 - x$ ,  $\eta = 1 - y$ ,  $\alpha = -x/\xi$  and  $\beta = -y/\eta$ .

Putting  $x = -\exp(i\pi/3)$  and  $y = \exp(i\pi/3)$  in Eq. (65) and simplifying, gives

$$\begin{aligned}
& \frac{1}{64} \pi^2 \ln^3 3 - \frac{127}{3456} \pi^4 \ln 3 - \frac{1}{128} \ln^5 3 + \frac{1573}{144} \zeta(5) \\
& = \frac{3}{2} \text{Li}_5 \left[ -\frac{1}{3} \right] - 3 \text{Li}_5 \left[ \frac{1}{3} \right] + 9 \text{Li}_5 \left[ \frac{1}{\sqrt{3}} e^{i\pi/6} \right] + 9 \text{Li}_5 \left[ \frac{1}{\sqrt{3}} e^{-i\pi/6} \right].
\end{aligned} \tag{66}$$

On taking real parts

$$\begin{aligned}
& \frac{1}{64} \pi^2 \ln^3 3 - \frac{127}{3456} \pi^4 \ln 3 - \frac{1}{128} \ln^5 3 + \frac{1573}{144} \zeta(5) \\
& = 18 \text{Re Li}_5 \left[ \frac{1}{\sqrt{3}} e^{i\pi/6} \right] + \frac{3}{2} \text{Li}_5 \left[ -\frac{1}{3} \right] - 3 \text{Li}_5 \left[ \frac{1}{3} \right].
\end{aligned} \tag{67}$$

Now

$$\begin{aligned}
& 18 \text{Re Li}_5 \left[ \frac{1}{\sqrt{3}} e^{i\pi/6} \right] = 18 \sum_{k=0}^{\infty} \left( \frac{1}{\sqrt{3}} \right)^k \frac{1}{k^5} \cos \left( \frac{k\pi}{6} \right) \\
& = \frac{1}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^8}{(12k+1)^5} + \frac{3^7}{(12k+2)^5} - \frac{3^6}{(12k+4)^5} \right. \\
& \quad - \frac{3^6}{(12k+5)^5} - \frac{2 \cdot 3^5}{(12k+6)^5} - \frac{3^5}{(12k+7)^5} - \frac{3^4}{(12k+8)^5} \\
& \quad \left. + \frac{3^3}{(12k+10)^5} + \frac{3^3}{(12k+11)^5} + \frac{2 \cdot 3^2}{(12k+12)^5} \right],
\end{aligned} \tag{68}$$

$$\begin{aligned}
\frac{3}{2} \operatorname{Li}_5 \left[ -\frac{1}{3} \right] &= \frac{3}{2} \sum_{k=1}^{\infty} \left( -\frac{1}{3} \right)^k \frac{1}{k^5} \\
&= \frac{1}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{-3^6 \cdot 2^4}{(12k+2)^5} + \frac{3^5 \cdot 2^4}{(12k+4)^5} - \frac{3^4 \cdot 2^4}{(12k+6)^5} \right. \\
&\quad \left. + \frac{3^3 \cdot 2^4}{(12k+8)^5} - \frac{3^2 \cdot 2^4}{(12k+10)^5} + \frac{3 \cdot 2^4}{(12k+12)^5} \right]
\end{aligned} \tag{69}$$

and

$$\begin{aligned}
3 \operatorname{Li}_5 \left[ \frac{1}{3} \right] &= 3 \sum_{k=1}^{\infty} \left( \frac{1}{3} \right)^k \frac{1}{k^5} \\
&= \frac{1}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^6 \cdot 2^5}{(12k+2)^5} + \frac{3^5 \cdot 2^5}{(12k+4)^5} + \frac{3^4 \cdot 2^5}{(12k+6)^5} \right. \\
&\quad \left. + \frac{3^3 \cdot 2^5}{(12k+8)^5} + \frac{3^2 \cdot 2^5}{(12k+10)^5} + \frac{3 \cdot 2^5}{(12k+12)^5} \right].
\end{aligned} \tag{70}$$

Using Eq. (68), (69) and (70) in Eq. (67), we obtain the ternary BBP-type formula

$$\begin{aligned}
&\frac{1}{64} \pi^2 \ln^3 3 - \frac{127}{3456} \pi^4 \ln 3 - \frac{1}{128} \ln^5 3 + \frac{1573}{144} \zeta(5) \\
&= \frac{1}{3^5} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[ \frac{3^7}{(12k+1)^5} - \frac{5 \cdot 3^7}{(12k+2)^5} - \frac{19 \cdot 3^4}{(12k+4)^5} \right. \\
&\quad - \frac{3^5}{(12k+5)^5} - \frac{2 \cdot 3^6}{(12k+6)^5} - \frac{3^4}{(12k+7)^5} \\
&\quad \left. - \frac{19 \cdot 3^2}{(12k+8)^5} - \frac{5 \cdot 3^3}{(12k+10)^5} + \frac{3^2}{(12k+11)^5} - \frac{10}{(12k+12)^5} \right].
\end{aligned} \tag{71}$$

## 7 Conclusion

Using fairly straightforward methods, we have obtained several ternary BBP-type formulas, which can now be added to the literature. In particular we proved the following formulas (written in the now standard BBP notation [3]).

$$\ln 2 = 1/(2 \cdot 3^5)P((1, 3^6, 12, (3^5, 3^5, 0, -3^4, -3^3, 0, -3^2, -3^2, 0, 3, 1, 0)))$$

$$\ln 3 = 1/3^6P((1, 3^6, 12, (3^6, 3^5, 0, -3^4, -3^4, -2 \cdot 3^3, -3^3, -3^2, 0, 3, 3, 2)))$$

$$\ln 3 = 1/27P((1, -27, 6, (27, 9, 0, -3, -3, -2)))$$

$$\pi\sqrt{3} = 1/3^4P((1, 3^6, 12, (3^5, 3^5, 2 \cdot 3^4, 3^4, 3^3, 0, -3^2, -3^2, -6, -3, -1, 0)))$$

$$\pi\sqrt{3} = 2/3^4P((1, 3^6, 12, (3^5, 0, -3^4, 0, 3^3, 0, -3^2, 0, 3, 0, -1, 0)))$$

$$\pi\sqrt{3} = 6P((1, -3, 2, (1, 0)))$$

$$\pi^2 = 2/27P((2, 3^6, 12, (3^5, -5 \cdot 3^4, 0, -3^4, -3^3, -2^3 \cdot 3^2, -3^2, -3^2, 0, -5, 1, 0)))$$

$$\ln^2 3 = 1/3^6P((2, 3^6, 12, (2 \cdot 3^7, -2 \cdot 3^8, 0, -2 \cdot 13 \cdot 3^4, -2 \cdot 3^5, -2^3 \cdot 3^5, -2 \cdot 3^4, -2 \cdot 13 \cdot 3^2, 0, -2 \cdot 3^4, 2 \cdot 3^2, -8)))$$

$$\pi\sqrt{3} \ln 3 = 2P((2, -27, 6, (9, -15, -18, -5, 1, 0)))$$

$$\sqrt{3}\text{Cl}_2(\pi/3) = 1/3P((2, -27, 6, (9, -9, -12, -3, 1, 0)))$$

$$13\zeta(3) - \pi^2 \ln 3 + \ln^3 3 = 2/3^5P((3, 3^6, 6, (3^6, 3^4, 3^4, 3^2, 3^2, 1)))$$

$$13\zeta(3) - \pi^2 \ln 3 + \ln^3 3 = 2/3P((3, 9, 2, (9, 1)))$$

$$(29\pi^3/1296 - \pi \ln^2 3/48)/\sqrt{3} = 1/2/3^5P((3, 3^6, 12, (3^6, -5 \cdot 3^5, -2 \cdot 3^6, -5 \cdot 3^4, 3^4, 0, -3^3, 5 \cdot 3^2, 2 \cdot 3^3, 5 \cdot 3, -3, 0)))$$

$$127\pi^4/5184 - \pi^2 \ln^2 3/32 + 5 \ln^4 3/192 = 1/3^6P((4, 3^6, 12, (3^7, -5 \cdot 3^7, 0, -19 \cdot 3^4, -3^5, -2 \cdot 3^6, -3^4, -19 \cdot 3^2, 0, -5 \cdot 3^3, 3^2, -10)))$$

$$(11\text{Cl}_4(\pi/3) - 29\pi^3 \ln 3/288 + \pi \ln^3 3/32)/\sqrt{3} = 1/2P((4, -27, 6, (9, -15, -18, -5, 1, 0)))$$

$$\pi^2 \ln^3 3/64 - 127\pi^4 \ln 3/3456 - \ln^5 3/128 + 1573\zeta(5)/144 = 1/3^5P((5, 3^6, 12, (3^7, -5 \cdot 3^7, 0, -19 \cdot 3^4, -3^5, -2 \cdot 3^6, -3^4, -19 \cdot 3^2, 0, -5 \cdot 3^3, 3^2, -10)))$$

We also proved the following ternary Zero Relations:

$$0 = P((1, 3^6, 12, (3^5, -5 \cdot 3^4, 0, -3^4, -3^3, -2^3 \cdot 3^2, -3^2, -3^2, 0, -5, 1, 0)))$$

$$0 = P((1, 3^6, 12, (3^5, -3^5, -2^2 \cdot 3^4, -3^4, 3^3, 0, -3^2, 3^2, 3 \cdot 2^2, 3, -1, 0)))$$

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## References

- [1] David H. Bailey, Peter B. Borwein, and Simon Plouffe. On the rapid computation of various polylogarithmic constants. *Mathematics of Computation*, 66(218):903–913, 1997.
- [2] D. H. Bailey. A compendium of bbp-type formulas for mathematical constants <http://crd.lbl.gov/~dhbailey/dhbpapers/bbp-formulas.pdf>. October 2010.
- [3] David H. Bailey and Richard E. Crandall. On the random character of fundamental constant expansions. *Experimental Mathematics*, 10:175, 2001.



- [4] Jonathan M. Borwein, David Borwein, and William F. Galway. Finding and excluding  $b$ -ary machin-type BBP formulae. *Available Online* <http://crd.lbl.gov/~dhbailey/dhbpapers/machin.pdf>, 2002.
- [5] Marc Chamberland. Binary bbp-formulae for logarithms and generalized gaussian-mersenne primes. *Journal of Integer Sequences*, 6, 2003.
- [6] Leonard Lewin. *Polylogarithms and associated functions*. Elsevier North Holland Inc., 1981.
- [7] Kunle Adegoke. A non-pslq route to bbp-type formulas. *Journ. Maths. Res.*, 2:56–64, 2010.
- [8] Kunle Adegoke. New binary and ternary digit extraction (bbp-type) formulas for trilogarithm constants. *New York J. Math.*, 16:361–367, 2010.
- [9] David J. Broadhurst. Massive 3-loop Feynman Diagrams Reducible to SC\* Primitives of algebras of the sixth root of unity. *Available Online* <http://xxx.lanl.gov/format/hep-ph/9803091>, 1998.
- [10] D. J. Broadhurst. Polylogarithmic ladders, hypergeometric series and the ten millionth digits of  $\zeta(3)$  and  $\zeta(5)$ . *arXiv:math/9803067v1 [math.CA]*, 1998.